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Algebraic Complexity Theory

SS 2014, Exercise Sheet #12

EXERCISE 23:

a) For a term $t(X_1,\ldots,X_n)$ and a field $\mathbb{F}\subseteq\mathbb{C}$ let

$$\mathrm{maxdim}(t,d) \;:=\; \max\big\{\dim\big(t(x_1,\ldots,x_n)\big) : x_1,\ldots,x_n\in\mathrm{Gr}(\mathbb{F}^d)\big\} \quad .$$

Prove that, for terms $s(X_1,\ldots,X_n)$ and $t(Y_1,\ldots,Y_m)$,
it holds $\mathrm{maxdim}_d(s\vee t)=\min\big\{\mathrm{maxdim}(s,d)+\mathrm{maxdim}(t,d),d\big\}$.
In particular $t(\bar{X})$ is weakly satisfiable over $\mathrm{Gr}(V)$ with $d=\dim(V)$
iff $t(\bar{X}^{(1)})\vee t(\bar{X}^{(2)})\vee\ldots\vee t(\bar{X}^{(d)})$ is strongly satisfiable over $\mathrm{Gr}(V)$.

b) For terms $s(X_1,\ldots,X_n)=s(\bar{X})$ and $t(\bar{Y})$ abbreviate

$$\big(s\big|_t\big)(\bar{X},\bar{Y}) \quad := \quad s\big(X_1\wedge t(\bar{Y}),\ldots,X_n\wedge t(\bar{Y})\big)\wedge t(\bar{Y}) \quad .$$

Prove $\mathrm{maxdim}(s|_t,d)=\mathrm{maxdim}\big(s,\mathrm{maxdim}(t,d)\big)$.
Do you need some hypothesis on $\mathbb{F}$?

c) Construct a term t_n of length $\mathcal{O}(n)$ strongly satisfiable over $\mathrm{Gr}(\mathbb{F}^{2^n})$ but not over $\mathrm{Gr}(\mathbb{F}^{2^n-1})$.
Hint: Iterate Exercise 22e)

d) Let $t(X_1,\ldots,X_n)$ denote a lattice (!) term of length $|t|$ and $z\in\mathrm{Gr}_1(V)$ and $x_1,\ldots,x_n\in\mathrm{Gr}(V)$ with $z\leq t(x_1,\ldots,x_n)$. Prove by induction on $|t|$:
There exist $y_j\leq x_j$ with $\dim(y_1)+\ldots+\dim(y_n)\leq|t|$ and $z\leq t(y_1,\ldots,y_n)$.

e) Show that to V there exists $n\in\mathbb{N}$ and terms $s(X;Y_1,\ldots,Y_n)$ and $t(X;Y_1,\ldots,Y_n)$ such that for every $x\in\mathrm{Gr}(V)$ it holds

$$x\neq\mathbf{0} \;\Leftrightarrow\; \exists y_1,\ldots,y_n\in\mathrm{Gr}(V):\ s(x,y_1,\ldots,y_n)=\mathbf{1} \qquad \text{and}$$
$$x=\mathbf{1} \;\Leftrightarrow\; \exists y_1,\ldots,y_n\in\mathrm{Gr}(V):\ t(x,y_1,\ldots,y_n)\neq\mathbf{0}$$