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Algebraic Complexity Theory

SS 2014, Exercise Sheet #11

EXERCISE 19:
Describe a BCSS machine (over $\mathbb{R}$ without constants) computing,

a) given $A,B \in \mathbb{R}^{d\times d}$, some $C \in \mathbb{R}^{d\times d}$ with $\mathrm{range}(C) = \mathrm{range}(A) \vee \mathrm{range}(B)$.

c) given $A \in \mathbb{R}^{d\times d}$, some $C \in \mathbb{R}^{d\times d}$ with $\mathrm{range}(C) = \neg\,\mathrm{range}(A)$.

b) given $A,B \in \mathbb{R}^{d\times d}$, some $C \in \mathbb{R}^{d\times d}$ with $\mathrm{range}(C) = \mathrm{range}(A) \wedge \mathrm{range}(B)$.

What is the asymptotic running time? Can we replace $\mathbb{R}$ with $\mathbb{C}$?

EXERCISE 22:

a) Suppose $\varphi : V \to W$ is an isomorphism of vector spaces and abbreviate $\varphi[U] := \{\varphi(\vec{u}) : \vec{u} \in U\}$. Show that $\varphi[X \vee Y] = \varphi[X] \vee \varphi[Y]$ and $\varphi[X \wedge Y] = \varphi[X] \wedge \varphi[Y]$ for all $X,Y \in \mathrm{Gr}(V)$.

b) Suppose φ is in addition an isometry of inner product spaces. Show that $\varphi[\neg X] = \neg\varphi[X]$.

c) Every term $t(X_1,\ldots,X_n)$ can be written as $s(X_1,\ldots,X_n,\neg X_1,\ldots,\neg X_n)$ for a lattice term s.

d) Prove: If t is strongly satisfiable over $\mathrm{Gr}(V)$ and over $\mathrm{Gr}(W)$, then t is also strongly satisfiable over $\mathrm{Gr}(V\times W)$. If t is weakly satisfiable over $\mathrm{Gr}(V)$ and V is a subspace of W, then t is also weakly satisfiable over $\mathrm{Gr}(W)$. Hint: $t_V(x_1,\ldots,x_n) = t_W(x_1,\ldots,x_n) \cap V$ for $x_1,\ldots,x_n \in \mathrm{Gr}(V)$.

e) Show that $x \vee \neg y = \mathbf{1}$ for $x,y \in \mathrm{Gr}(V)$ implies $\dim(x) \geq \dim(y)$.

f) Conclude that the following term h_d of length $\mathcal{O}(d^2)$ is strongly satisfiable over $\mathrm{Gr}(V)$ iff $d\,|\dim(V)$:

$$\Big(\bigvee_{j=1}^{d} X_j\Big) \;\wedge\; \Big(\bigwedge_{i\neq j} X_j \vee \neg X_i\Big) \;\wedge\; \Big(\bigwedge_{j=1}^{d} \neg\big(X_j \wedge \bigvee_{i\neq j} X_i\big)\Big)$$

and any satisfying assignment $x_1,\ldots,x_n \in \mathrm{Gr}(V)$ has $\dim(x_1) = \ldots = \dim(x_n) = \dim(V)/d$.

g) Verify that $D_j := \mathbb{F}\vec{e}_j$ and $D_0 := \neg\mathbb{F}(\vec{e}_1 + \ldots + \vec{e}_d)$ constitute a d–*diamond* (see the script). Prove that any d–diamond $D_0,D_1,\ldots,D_d$ has $\dim(V) - \dim(D_0) = \dim(D_1) = \ldots = \dim(D_d) = \dim(V)/d$ and weakly satisfies the following term $g_d(Z_0,Z_1,\ldots,Z_d) = g_d(\bar{Z})$:

$$\neg Z_0 \wedge \bigwedge_{j=1}^{d} \big(Z_0 \vee g_{d,j}(\bar{Z})\big), \qquad \text{where } g_{d,j}(\bar{Z}) := Z_j \wedge \bigwedge_{i\neq j>0} \neg Z_i$$