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Algebraic Complexity Theory

SS 2014, Exercise Sheet #10

EXERCISE 19:
For a vector space V, the Grassmannian $\mathrm{Gr}_k(V)$ is the set of k–dimensional linear subspaces of V; $\mathrm{Gr}(V) := \bigcup_k \mathrm{Gr}_k(V)$, $\mathbf{0} := \{\vec{0}\}$, $\mathbf{1} := V$. Equip $\mathrm{Gr}(V)$ with the operations

$$X \wedge Y := X \cap Y, \quad X \vee Y := X + Y, \quad \text{and } \neg X := X^{\perp} = \{\vec{v} \in V : \forall \vec{a} \in X : \vec{v} \perp \vec{a}\}$$

supposing V comes with an inner product to denote orthogonality $\perp$.

a) Show that $\mathrm{Gr}(V)$ satisfies the de Morgan rules:
$\neg(X \vee Y) = (\neg X) \wedge (\neg Y)$ and $\neg(X \wedge Y) = (\neg X) \vee (\neg Y)$.

b) Show that $\mathrm{Gr}(\mathbb{R}^2)$ violates the distributive law: $(X \vee Y) \wedge Z \neq (X \wedge Z) \vee (Y \wedge Z)$.
Moreover is satisfies the *disjunction property* for weak truth but not for strong truth:

$$X \vee Y \neq \mathbf{0} \Rightarrow X \neq \mathbf{0} \vee Y \neq \mathbf{0}, \qquad X \vee Y = \mathbf{1} \not\Rightarrow X = \mathbf{1} \vee Y = \mathbf{1} \ .$$

c) $C(X,Y) := (X \wedge Y) \vee (X \wedge \neg Y) \vee (\neg X \wedge Y) \vee (\neg X \wedge \neg Y)$ is called the *commutator* of X and Y.
Evaluate $C(X,Y)$ on $\mathrm{Gr}(\mathbb{F}^1)$ and on $\mathrm{Gr}(\mathbb{R}^2)$ — in a systematic way.

d) Show that for arbitrary $X, Y \in \mathrm{Gr}(V)$ it holds $X = Y \Leftrightarrow (X \wedge Y) \vee (\neg X \wedge \neg Y) = \mathbf{1}$.

EXERCISE 20:
Describe a BCSS machine (over $\mathbb{R}$ without constants) computing,

a) given $A, B \in \mathbb{R}^{d \times d}$, some $C \in \mathbb{R}^{d \times d}$ with $\mathrm{range}(C) = \mathrm{range}(A) \vee \mathrm{range}(B)$.

b) given $A, B \in \mathbb{R}^{d \times d}$, some $C \in \mathbb{R}^{d \times d}$ with $\mathrm{range}(C) = \mathrm{range}(A) \wedge \mathrm{range}(B)$.

c) given $A \in \mathbb{R}^{d \times d}$, some $C \in \mathbb{R}^{d \times d}$ with $\mathrm{range}(C) = \neg\,\mathrm{range}(A)$.

Can we replace $\mathbb{R}$ with $\mathbb{C}$?

EXERCISE 21:
To every $N \in \mathbb{Z}$ there exists a term t over $(\times, -, 1)$ of length $\mathcal{O}(\log N)$ that over $\mathbb{Z}$ evaluates to N.