Martin Ziegler

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Algebraic Complexity Theory

SS 2014, Exercise Sheet #9

EXERCISE 17:
Referring to Definition 6.7 in the script:

a) Prove that $\mathsf{XNONTRIV}^0_{\mathbb{P}^2(\mathbb{R})}$ and $\mathsf{XNONTRIV}^0_{\mathbb{R}^3}$ and $\mathsf{XUVEC}^0_{\mathbb{R}^3}$ are equal as sets.

b) Construct polynomial-time reductions from $\mathsf{XNONEQUIV}^0_{\mathbb{P}^2(\mathbb{R})}$ to $\mathsf{XNONTRIV}^0_{\mathbb{R}^3}$ and back. (Hint: $[\vec{s}] \neq [\vec{t}] \Leftrightarrow \vec{s}\times\vec{t} \neq \vec{0}$).

c) Construct a polynomial-time reduction from $\mathsf{XNONTRIV}^0_{\mathbb{R}^3}$ to Polynomial Identity Testing.

d) Verify that $\vec{t}' := (\vec{t}\times\vec{w})\times\big((\vec{t}\times\vec{w})\times\vec{t}\big)$ is a multiple of $\vec{t}$ for any $\vec{w}$ non-parallel to $\vec{t}$. Can every multiple of $\vec{t}$ be obtained in this way?

e) Prove that $\vec{s}' := \Big(\big(\vec{w}\times(\vec{s}\times\vec{w})\big)\times\vec{s}\Big)\times\big(\vec{s}\times(\vec{s}\times\vec{w})\big)$ is a multiple of $\vec{s}$; and every multiple of $\vec{s}$ can be obtained in this way for some appropriate $\vec{w}$.

f) Construct a polynomial-time reduction from $\mathsf{XSAT}^0_{\mathbb{P}^2(\mathbb{R})}$ to $\mathsf{XSAT}^0_{\mathbb{R}^3}$.

EXERCISE 18:
Fix a subfield $\mathbb{F}$ of $\mathbb{R}$, such as $\mathbb{Q}$ and recall that $\mathbb{P}^2(\mathbb{F}) = \{\ [\vec{v}] : \vec{0} \neq \vec{v} \in \mathbb{F}^3\}$ denotes the projective space over $\mathbb{F}^3$, where $[\vec{v}] := \{\lambda\vec{v} : \lambda \in \mathbb{F}\}$. To $A,B,C \in \mathbb{P}^2(\mathbb{F})$ consider

$$V_{12} := B \quad V_2 := (A\times B)\times A \quad V_{23} := C\times A \quad V_1 := V_2\times V_{23} \quad V_3 := \big(V_{23}\times(B\times V_2)\big)\times B \qquad (1)$$

a) The evaluation of these terms may be undefined for some assignments of A,B,C. Verify that, on the other hand, $A := [\vec{v}_1]$, $B := [\vec{v}_2 - \vec{v}_1]$ and $C := [\vec{v}_2 + \vec{v}_3]$, do evaluate — namely to

$$V_1 = [\vec{v}_1], \quad V_2 = [\vec{v}_2], \quad V_3 = [\vec{v}_3], \quad V_{12} = [\vec{v}_1 - \vec{v}_2], \quad V_{23} = [\vec{v}_2 - \vec{v}_3]$$

— for any choice of pairwise orthogonal non-zero $\vec{v}_1,\vec{v}_2,\vec{v}_3 \in \mathbb{F}^3$.

b) Conversely when all quantities in Equation (1) are defined, then $V_1 = A$ and there exists a right-handed (!) orthogonal basis $\vec{v}_1,\vec{v}_2,\vec{v}_3$ of $\mathbb{F}^3$ such that $V_j = [\vec{v}_j]$ and $V_{12} = [\vec{v}_1 - \vec{v}_2]$ and $V_{23} = [\vec{v}_2 - \vec{v}_3]$.