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Issued on 2014-06-10
Solutions due: 2014-06-16

Algebraic Complexity Theory

SS 2014, Exercise Sheet #8

EXERCISE 14:

a) Devise a BCSS machine over $\mathbb{R}$ computing the algebraic degree function $\deg : \mathbb{A} \to \mathbb{N}$.

b) Prove that the closure, interior, and boundary of a semi-algebraic set are again semi-algebraic. (Hint: Tarski–Seidenberg)

c) Prove that every non-empty semi-algebraic set contains a (componentwise) algebraic point.

EXERCISE 15:
A *perfect matching* in a bipartite graph $G = (U \uplus V, E)$ is a subset $F \subseteq E$ of its edges $E \subseteq U \times V$ such that every vertex $u \in U$ is incident to exactly one edge in F and similarly for every $v \in V$.

a) W.l.o.g. suppose $U = \{1, \ldots, n\}$ and $V = \{-1, \ldots, -n\}$. Prove that G has a perfect matching iff the following polynomial in n^2 variables $X_{u,v}$, $(u \in U, v \in V)$, is non-zero:

$$\sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \cdot \prod_{u=1}^{n} \begin{cases} X_{u,-\sigma(u)} & \text{if } \big(u, -\sigma(u)\big) \in E \\ 0 & \text{if } \big(u, -\sigma(u)\big) \notin E \end{cases}$$

b) Devise and analyze a randomized polynomial-time algorithm for checking whether a given bipartite graph on $2n$ vertices contains a perfect matching.
Try to make it err with probability less than $1/2^n$.

EXERCISE 16:
Recall the 3D cross product and show that the term $\Big(\big(V \times (V \times W)\big) \times V\Big) \times (V \times W)$ always evaluates to the zero vector.