Martin Ziegler

Issued on 2014-05-27
Solutions due: 2014-06-02

# Algebraic Complexity Theory

## SS 2014, Exercise Sheet #6

**EXERCISE 11:**
Devise straight-line programs over $\mathcal{S}_1 = \big(\mathbb{C},\mathbb{C},+,\times,\div\big)$ of length $\mathcal{O}(N\cdot\operatorname{polylog}N)$ for multiplying the following structured $N\times N$–matrices, given by their $\mathcal{O}(N)$ parameters, to a given $N$–vector:

a) A circulant matrix $C(c_0,\dots,c_{N-1}) := \big(c_{j-i \bmod N}\big)_{0\le i,j<N}$

b) A Toeplitz matrix $T(t_{-N+1},\dots,t_0,\dots,t_{N-1}) := \big(t_{j-i}\big)_{0\le i,j<N}$

c) A Hankel matrix $H(h_{-N+1},\dots,h_0,\dots,h_{N-1}) := \big(h_{j+i-N}\big)_{0\le j<N;1\le i\le N}$

d) A Vandermonde matrix $V(v_1,\dots,v_N) := \big(v_i^j\big)_{1\le i\le N;0\le j<N}$

e) A generalized Hilbert matrix $G_p(z_1,\dots,z_N;w_1,\dots,w_N) := \big((z_j-w_i)^{-p}\big)_{1\le i,j\le N}$
where $p\in\mathbb{N}$ is fixed and $z_j\neq w_i$. (For $p=1$ this is known as *Trummer's Problem*…)

$$\begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{N-2} & c_{N-1}\\ c_{N-1} & c_0 & c_1 & c_2 & \ddots & c_{N-2}\\ c_{N-2} & c_{N-1} & c_0 & c_1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & c_2\\ c_2 & c_3 & \ddots & c_{N-1} & c_0 & c_1\\ c_1 & c_2 & \dots & c_{N-2} & c_{N-1} & c_0 \end{pmatrix}, \qquad \begin{pmatrix} h_{-N+1} & h_{-N+2} & \dots & h_{-2} & h_{-1} & h_0\\ h_{-N+2} & h_{-N+3} & \dots & h_{-1} & h_0 & h_1\\ h_{-N+3} & h_{-N+4} & \dots & h_0 & h_1 & h_2\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ h_{-2} & h_{-1} & h_0 & h_1 & \ddots & \vdots\\ h_{-1} & h_0 & h_1 & h_2 & \dots & h_{N-2}\\ h_0 & h_1 & h_2 & \dots & h_{N-2} & h_{N-1} \end{pmatrix}$$

$$\begin{pmatrix} t_0 & t_1 & t_2 & \dots & t_{N-2} & t_{N-1}\\ t_{-1} & t_0 & t_1 & t_2 & \dots & t_{N-2}\\ t_{-2} & t_{-1} & t_0 & t_1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ t_{-N+2} & t_{-N+3} & \dots & t_{-1} & t_0 & t_1\\ t_{-N+1} & t_{-N+2} & \dots & t_{-2} & t_{-1} & t_0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & v_1 & v_1^2 & \cdots & v_1^{N-1}\\ 1 & v_2 & v_2^2 & \cdots & v_2^{N-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & v_N & v_N^2 & \cdots & v_N^{N-1} \end{pmatrix}$$

**EXERCISE 12:**
The 2D Gravitation/Coulomb potential $u$ of a charge/mass distribution $f:\mathbb{R}^2\to\mathbb{R}$ satisfies Poisson's equation $\Delta u = f$.

a) Verify that the *fundamental solution* $u(x,y)=\ln(x^2+y^2)/2$ is a solution on $\mathbb{R}^2\setminus\{(0,0)\}$ with 'force' $\nabla u(x,y) = (\mathrm{Re},\mathrm{Im})\frac{1}{x-iy}$.

b) Devise a subquadratic-time algorithm for computing, given the coordinates $(x_n,y_n)$ of $N$ particles in the plane, their $N$ mutual forces $\sum_{1\le m\neq n}^{N}\nabla u(x_n-x_m,y_n-y_m)$, $1\le n\le N$.