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Complexity Theory

WS 2011/2012, Exercise Sheet #9

EXERCISE 24:
The lecture has considered Boolean circuits composed from binary `AND` and `OR` and unary `NOT`.

a) Let us now temporarily permit `AND` and `OR` gates with an unbounded number of inputs, the *size* of such a gate being this number. Prove: In every circuit of polynomial (why?) size, these gates can be replaced by their binary counterparts while the size grows at most by a constant factor and depth by at most a logarithmic one.

b) Now demonstrate that the three above types of gates can be replaced by one single type to rebuild all circuits while affecting both depth and size by no more than a constant factor.

c) Prove that, from `AND` and `OR` alone, only *monotone* functions can be realized, i.e. satisfying

$$\forall 1 \leq j \leq n \quad \forall x_1, \ldots, x_n : \quad f(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) \leq f(x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n) \ .$$

d) Recall Exercise 9g) and prove: Every language $L \subseteq \{0,1\}^*$ can be decided by a family of circuits of circuits. What size and depth do you achieve? Is the family uniform?

e) The Boolean function $\mathrm{Mux}_n : (x_1, \ldots, x_n; y_0, \ldots, y_{2^n-1}) \mapsto y_{(x_1,\ldots,x_n)_2}$ is called a Multiplexer. Sketch a circuit C_n for Mux_n of depth $\mathcal{O}(n)$. Is $(C_n)_n$ uniform?

EXERCISE 25:
Fix $\Gamma \subseteq \{0,1\}^k$ and $Q \subseteq \{0,1\}^\ell$ and $\delta : \Gamma \times Q \mapsto \Gamma \times Q \times \{00, 01, 10\}$.

a) Describe a circuit C_δ computing δ.

b) Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta)$ denote a DTM and $s \in \mathbb{N}$. Describe a circuit $C'_{\delta,s}$ with the following functionality:
Its input consists of the encoding of a configuration of $\mathcal{M}$: state, tape contents of length $\leq s$ with one bit per cell indicating whether the head is presently reading it. The output of $C'_{\delta,s}$ the encodes $\mathcal{M}$'s configuration one step later, that is, the new state, new tape contents with updated indicators for the head position.
How large and deep is $C'_{\delta,s}$, depending on s?

c) Let $n \in \mathbb{N}$, $s \geq S_{\mathcal{M}}(n)$, and $t \geq T_{\mathcal{M}}(n)$. Describe a circuit $C'_{\mathcal{M},s,t}$ simulating $\mathcal{M}$ on all inputs of length n. Estimate the size and depth of $C'_{\delta,s,t}$ asymptotically in terms of s and t.

d) Prove that a polynomial-time DTM $\mathcal{M}$ can be simulated by a uniform family of polynomial size circuits. Moreover, CIRCUITVAL is $\mathcal{P}$-complete.