Martin Ziegler
Carsten Rösnick

Distributed on 24.11.2011
Solutions to be submitted by 1.12.2011 14h25

Complexity Theory

WS 2011/2012, Exercise Sheet #5

Knapsack is the problem of collecting packets to achieve large 'value' at small 'weight'. Its decision variant is the language

$$\text{KNAPSACK} = \Big\{\langle g_1,\ldots,g_m,g,w_1,\ldots,w_m,w\rangle : m,g_i,g,w_i,w \in \mathbb{N}, \\ \exists \alpha_1,\ldots,\alpha_m \in \{0,1\} : \textstyle\sum_i \alpha_i w_i \geq w \,\wedge\, \sum_i \alpha_i g_i \leq g\Big\}$$

Its maximization variant is the function

$$\text{MAXKNAPSACK} : \langle g_1,\ldots,g_m,g,w_1,\ldots,w_m\rangle \mapsto \max\Big\{\textstyle\sum_i \alpha_i w_i : \alpha_i \in \{0,1\}, \sum_i \alpha_i g_i \leq g\Big\}$$

Integer linear programming is the decision problem

$$\text{ILP} := \Big\{\langle A,b\rangle : m \in \mathbb{N}, A \in \mathbb{Z}^{m\times m}, b \in \mathbb{Z}^m : \exists x \in \mathbb{Z}^m : A\cdot x \leq b\Big\} \ .$$

EXERCISE 13:

a) Prove that KNAPSACK is $\mathcal{NP}$-complete.

b) Prove: MAXKNAPSACK is computable in polynomial time iff KNAPSACK is decidable in polynomial time. Hint: Recall Exercise 7.

c) Prove that ILP is $\mathcal{NP}$-hard.

d) Reduce Hamilton Circuit HC to Traveling Salesperson TSP.

e) Let $\mathbb{P} = \{2,3,5,7,11,\ldots\}$ the set of prime numbers. Prove: $\mathbb{N}\setminus\mathbb{P} \in \mathcal{NP}$.

f) We consider the solvability problem over the integers of a multivariate polynomial equation:

$$\text{DIOPHANT} := \Big\{\langle p\rangle : m \in \mathbb{N}, p \in \mathbb{N}[X_1,\ldots,X_m], \exists x_1,\ldots,x_m \in \mathbb{N} : p(x_1,\ldots,x_m) = 0\Big\} \ .$$

Construct (and prove correctness of) an explicit polynomial-time reduction $\text{ILP} \preccurlyeq_p \text{DIOPHANT}$.

g) Are DIOPHANT and ILP $\mathcal{NP}$-complete? What makes this difficult to prove?

In a) and c) you may use that the following problems have already been shown $\mathcal{NP}$-complete in the lecture: SAT, 3SAT, SUBSETSUM.

EXERCISE 14:
Consider the following optimization problem: Given $\ell \in \mathbb{N}$ and m packets of weights $g_1,\ldots,g_m \in \mathbb{N}$; distribute them upon ℓ bins such as to minimize their maximal weight.

a) Formalize this as a decision problem and prove it in $\mathcal{NP}$.
(It has been shown $\mathcal{NP}$-hard even for $\ell = 2\ldots$)

b) Prove that the following greedy algorithm produces an approximation of rate 2:

Iteratively for each $i = 1,\ldots,m$ put packet #i into a currently lightest bin.

c) Demonstrate that the algorithm from b) achieves no better rate than 2 by constructing to $\varepsilon > 0$ an input on which its output exceeds the optimal solution by a factor at least $2-\varepsilon$.