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Complexity Theory

WS 2011/2012, Exercise Sheet #4

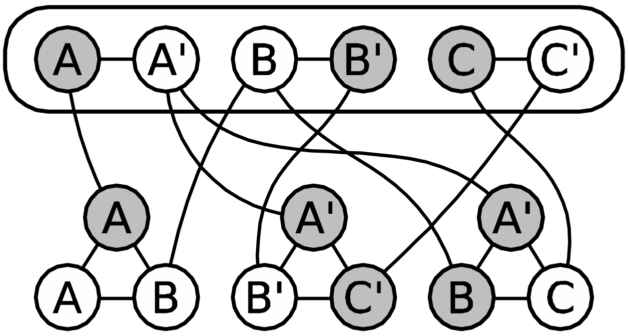
EXERCISE 10:
Recall the problems 3SAT and VC mentioned in the lecture.

a) Prove VC $\preccurlyeq_p$ SAT directly, i.e. without invoking the Cook-Levin Theorem.

b) Argue: every vertex cover of a 2-clique must contain at least one vertex; every vertex cover of a 3-clique must contain at least two vertices.

c) Prove: 3SAT $\preccurlyeq_p$ VC.

Hint: The following drawing illustrates a reduction mapping an instance $\Phi = (A \vee A \vee B) \wedge (\neg A \vee \neg B \vee \neg C) \wedge (\neg A \vee B \vee C)$ of 3SAT to an instance (G,k) of VC with $k :=$ #variables $+ 2 \cdot$ #clauses.



EXERCISE 11:
For a language $A \subseteq \Sigma^*$, its *Kleene-star* $A^* := \{\bar{a}_1\bar{a}_2\cdots\bar{a}_n | n \in \mathbb{N}_0, \bar{a}_i \in A\}$ is the language consisting of all concatenations of finitely many words from A.

a) Discuss and compare the two different meanings "Σ^*" can have. What is $\left(A^*\right)^*$?

b) Prove that $\mathcal{P}$ is closed under

   i) binary union, i.e., $A, B \in \mathcal{P} \Rightarrow A \cup B \in \mathcal{P}$
   ii) intersection, i.e. $A, B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P}$
   iii) and complement, i.e. $A \in \mathcal{P} \Rightarrow \Sigma^* \setminus A \in \mathcal{P}$.

   One can (but you don't need to) show that $\mathcal{P}$ is also closed under Kleene-star.

c) Prove PSPACE closed under i) union, ii) intersection, iii) complement, iv) Kleene-star.

d) Prove that also $\mathcal{NP}$ is closed under union, intersection, complement, and Kleene-star.

e) Demonstrate that the complements of languages in $\mathcal{NP}$ are precisely those of the form

$$\left\{\bar{x} \in \Sigma^* : \forall \bar{y} \in \Sigma^{\leq |\bar{x}|^k} : \langle \bar{x}, \bar{y} \rangle \in K\right\}, \qquad K \in \mathcal{P},\ k \in \mathbb{N}.$$

EXERCISE 12:
A *non*deterministic Turing maschine (NTM) $\mathcal{N} = (Q,\Sigma,\Gamma,\delta)$ has a transition *relation*

$$\Delta \quad \subseteq \quad \big((Q\setminus\{q_-,q_+\})\times\Gamma\big)\times\big(Q\times\Gamma\times\{\mathsf{L},\mathsf{N},\mathsf{R}\}\big) \ .$$

A transition $(q,a)\to(p,b,D)$ with $D\in\{\mathsf{L},\mathsf{R},\mathsf{N}\}$ is *valid* if $(q,a,p,b,D)\in\Delta$. A *computation* of $\mathcal{N}$ is a sequence of configurations such that each successor arises from its predecessor via a valid transition.
$\mathcal{N}$ *accepts* an input $\vec{w}$ if there *exists a* computation of $\mathcal{N}$ starting with configuration $(s,\vec{w})$ and leading to a configuration with accepting state.
$\mathcal{N}$ *accepts* the language $L\subseteq\Sigma^*$ if it accepts precisely those inputs from L. $\mathcal{N}$ *decides* L if, in addition, every computation of $\mathcal{N}$ terminates.
The run time $T_{\mathcal{N}}(\vec{w})$ on $\vec{w}$ is the maximum length of all computations of $\mathcal{N}$ on input $\vec{w}$; similarly for space $S_{\mathcal{N}}(\vec{w})$.

a) Explain why (and how) a DTM can be considered as a NTM.
Describe a NTM deciding Boolean satisfiability SAT within polynomial time.

b) Sketch the set of all possible computations of a NTM on fixed input $\vec{w}$ as a tree. Bound its degree: how many successors can each node have at most?
Argue that any NTM can be converted, possibly with a linear slow-down, into an equivalent one having at most two possible successor configurations in each step, i.e. such that $\mathrm{Card}\{(p,b,D):(q,a,p,b,D)\in\delta\}\leq 2$ holds for all $q\in Q$ and $a\in\Gamma$.

c) Construct to any (multi-head) NTM $\mathcal{N}$ a (multi-head) DTM $\mathcal{M}$ simulating $\mathcal{N}$ with

$$T_{\mathcal{M}}(n) \ \leq\ 2^{\mathcal{O}(T_{\mathcal{N}}(n))}, \qquad S_{\mathcal{M}}(n) \ \leq\ \mathcal{O}\big(T_{\mathcal{N}}(n)+S_{\mathcal{N}}(n)\big) \ .$$

d) Let $p\in\mathbb{N}[N]$ be a polynomial and $L\subseteq\Sigma^*$ decidable in deterministic time $\mathcal{O}\big(p(n)\big)$. Describe a NTM deciding L'_p in polynomial time, where

$$L'_p \ :=\ \big\{\vec{x}\in\Sigma^*:\exists\vec{y}\in\Sigma^{\leq p(|\vec{x}|)}:\langle\vec{x},\vec{y}\rangle\in L\big\} \ .$$

e) Conversely suppose K is decidable by a NTM in polynomial time. Prove that there exists $p\in\mathbb{N}[N]$ and a langauge L decidable in deterministic polynomial time such that $K=L'_p$.