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Complexity Theory

WS 2011/2012, Exercise Sheet #3

EXERCISE 9:
Let G denote a (directed or undirected) graph. Reachability is the question of whether, given G and a pair (u,v) of nodes, there exists a path from u to v in G.

a) Describe an algorithm deciding reachability in polynomial time.

Now consider 2SAT: the question of whether a Boolean term in two-conjunctive form (2CNF, i.e. with two literals per clause) admits a satisfying assignment.

b) Argue that the question of whether a given (reasonable binary encoding of a) Boolean term is in conjunctive form can be decided in polynomial time.

c) A Boolean term $t(x_1,\ldots,x_n)$ in in 2CNF induces a (so-called *implication*) graph G_t with literales $x_1,\ldots,x_n,\neg x_1,\ldots,\neg x_n$ as vertices and directed edges $(\neg u,v)$ and $(\neg v,u)$ between literales u,v for every clause $(u \vee v)$ of t. Prove: If t has a satisfying assignment and x is a variable of t, then G_t admits no path from x to $\neg x$ nor from $\neg x$ to x.

d) Suppose conversely for each variable x in t that G_t admits no path from x to $\neg x$ nor from $\neg x$ to x. Conclude that t has a satisfying assignment.

e) Prove that 2SAT is decidable in polynomial time.

f) A term $t = C_1 \vee C_2 \vee \cdots \vee C_m$ with each C_i a *con*junction of literales is called in *disjunctive* form. Show that satisfiability of terms in disjunctive form can be decided in polynomial time.

g) Prove: i) There are precisely 2^{2^n} distinct n-variate Boolean functions $f : \{0,1\}^n \to \{0,1\}$; and each of them can be expressed as a term in ii) disjunctive and iii) in conjunctive form.
Hint: Verify $$f(x_1,\ldots,x_n) \quad = \quad \bigvee_{\vec{y}:f(\vec{y})=1} \bigwedge_{i=1}^{n} \begin{cases} x_i & : y_i = 1 \\ \neg x_i & : y_i = 0 \end{cases}$$

h) Specify a family (f_n) of n-variate Boolean functions expressible as terms in disjunctive form of polynomial size but not as terms in conjunctive form of polynomial size (without proof).

j) Describe an algorithm transforming a given Boolean term t within polynomial time into a term s in conjunctive normal form such that it holds: t is satisfiable iff s is.

k) Consider the following problem called Equivalence:

Given two n-variate Boolean terms s and t, does it hold $\forall \vec{a} \in \{0,1\}^n : s(\vec{a}) = t(\vec{a})$?

Prove this problem as computationally as hard as *un*satisfiability, the complement of SAT.

ℓ) Prove or refute: i) $A \subseteq B \Rightarrow A \preccurlyeq_{\mathrm{p}} B$, ii) $A \preccurlyeq_{\mathrm{p}} B \Rightarrow A \subseteq B$.