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Complexity Theory

WS 2011/2012, Exercise Sheet #2

EXERCISE 6:
The Turing machines considered in the lecture employ one single head (accessing possibly multiple tapes simultaneously).

a) Formalize the concept of a Turing machine with k heads it can move independently. Where does its transition function map from/to?

b) In view of Exercise 4b) describe now a Turing machine with two heads deciding `PALIN` in linear time $\mathcal{O}(n)$.

EXERCISE 7:
So far we have considered decision problems (i.e. formal languages) $L \subseteq \Sigma^*$. A Turing machine $\mathcal{M}$ is said to compute a function $f : \text{dom}(f) \subseteq \Sigma^* \to \Sigma^*$ if

- on inputs $\vec{x} \in \text{dom}(f)$, $\mathcal{M}$ prints $f(\vec{x})$ and enters the accepting state;
- on inputs $\vec{x} \notin \text{dom}(f)$ does *not* enter the accepting state.

Running time and memory consumption are defined canonically. Prove:

a) If f can be computed in polynomial time, there exists a $k \in \mathbb{N}$ with $|f(\vec{x})| \leq \mathcal{O}(|\vec{x}|^k)$ for all $\vec{x} \in \text{dom}(f)$, where $|\vec{x}| = n$ denotes the length of $\vec{x} \in \Sigma^n$.

b) If f can be computed in polynomial time then the following language, subgraph(f), is polynomial-time decidable:

$$\big\{\langle\vec{x},\vec{y}\rangle : \vec{x} \in \text{dom}(f), \vec{y} \preccurlyeq f(\vec{x})\big\} \quad \text{where} \quad \vec{y} \preccurlyeq \vec{z} \;:\Leftrightarrow\; |\vec{y}| < |\vec{z}| \vee \big(|\vec{y}| = |\vec{z}| \wedge \text{bin}(\vec{y}) \leq \text{bin}(\vec{z})\big)$$

c) If subgraph(f) is decidable in polynomial time and $|f(\vec{x})| \leq \mathcal{O}(|\vec{x}|^k)$ holds for some $k \in \mathbb{N}$ and all $\vec{x} \in \text{dom}(f)$, then f can be computed in polynomial time.

EXERCISE 8:
Recall the algorithms discussed in the lecture for matrix multiplication in the so-called algebraic model: addition, multiplication, and division of arbitrary (nonzero) elements from the ground field are charged one step each. Fix $\omega \geq 2$ and prove:

a) If multiplication of upper triangular $n \times n$ matrices is possible in $\mathcal{O}(n^\omega)$ steps then so is multiplication of arbitrary $n \times n$-matrices.

b) If inverting non-singular upper triangular matrices is possible in $\mathcal{O}(n^\omega)$ steps then so is multiplication of arbitrary $n \times n$ matrices. Hint: Calculate the inverse of $\begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}$