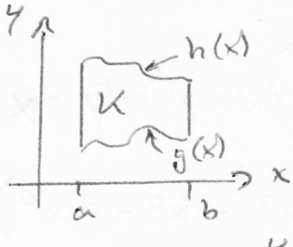
§ 5

5.1 Integrale im $\mathbb{R}^n$

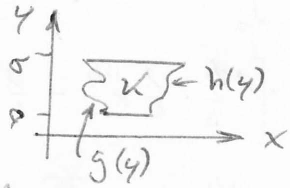
$K \subseteq \mathbb{R}^2$ kompakt:

y - projizierbar $:\Leftrightarrow$ Ex. $g, h : [a,b] \to \mathbb{R}$ stetig mit

$$K = \{(x,y) \in \mathbb{R}^2 \mid g(x) \le y \le h(x),\ x \in [a,b]\}$$

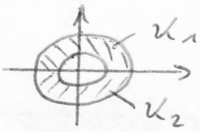


x - projizierbar: analog



Normalbereich: y- oder x-projizierbar.

$\to$ ggf. in projizierbare Mengen zerlegen



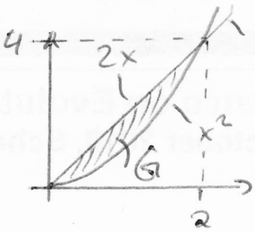
Aufgabe 1:

$$\int_R x^3 \sqrt{y}\, d(x,y) = \int_1^2 \int_0^4 x^3 \sqrt{y}\, dy\, dx = \int_1^2 x^3 \left[\tfrac{2}{3} y^{3/2}\right]_0^4 dx$$

$$= \frac{16}{3} \cdot \int_1^2 x^3\, dx = \frac{16}{3} \cdot \frac{1}{4} x^4 \Big|_1^2 = \frac{15}{4} \cdot \frac{16}{3} = 20$$

Integration über Rechteck $\to$ Integrationsreihenfolge vertauschbar (hier sogar "trennbar").

Aufgabe 2



G ist y-projizierbar:

$$G = \{(x,y) \in \mathbb{R}^2 \mid x^2 \le y \le 2x,\ x \in [0,2]\}$$

G ist x-projizierbar:

$$G = \{(x,y) \in \mathbb{R}^2 \mid \tfrac{y}{2} \le x \le \sqrt{y},\ y \in [0,4]\}.$$

$$\int_G (x^2+y^2)\, d(x,y) = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2+y^2)\, dx\, dy = \dots$$

oder:

$$\int_G (x^2+y^2)\, d(x,y) = \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\, dy\, dx = \int_0^2 \left[x^2 y + \tfrac{1}{3} y^3\right]_{x^2}^{2x} dx$$

$$= \int_0^2 \left(2x^3 + \tfrac{8}{3}x^3 - x^4 - \tfrac{1}{3}x^6\right) dx$$

$$= \left[\tfrac{14}{3} \cdot \tfrac{1}{4} x^4 - \tfrac{1}{5} x^5 - \tfrac{1}{3} \cdot \tfrac{1}{7} x^7\right]_0^2$$

$$= \tfrac{14}{3} \cdot 4 - \tfrac{32}{5} - \tfrac{128}{21}.$$

5.2 Substitutionsregel

$g: U \to V$ bijektiv, g, g^{-1} stetig differenzierbar

$f: K \to \mathbb{R}$ stetig, $K \subset U \subseteq \mathbb{R}^n$

$$\Rightarrow \int_K f(y_1,\dots,y_n)\, d(y_1,\dots,y_n) = \int_{g^{-1}(K)} f(g(x_1,\dots,x_n)) \cdot |\det J_g(x_1,\dots,x_n)|\, d(x_1,\dots,x_n)$$

wichtige Spezialfälle:

Polarkoordinaten im $\mathbb{R}^2$:

$x \to r\cos\varphi$

$y \to r \cdot \sin\varphi$

$$J_g(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$|\det J_g(r,\varphi)| = r$$

$$\Rightarrow dx\, dy = r\, d\varphi\, dr$$

$$\Rightarrow \int_K f(x,y)\, d(x,y) = \int_{g^{-1}(K)} f(r\cos\varphi, r\sin\varphi) \cdot r\, d(r,\varphi).$$

Kugelkoordinaten im $\mathbb{R}^3$:

$$\begin{matrix} x \to \\ y \to \\ z \to \end{matrix} \begin{pmatrix} r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta \\ r\sin\theta \end{pmatrix} =: g(r,\varphi,\theta)$$

$$g:]0,\infty[\times]0,2\pi[\times]-\tfrac{\pi}{2},\tfrac{\pi}{2}[\to \mathbb{R}^3$$

Dabei ist $|\det J_g(r,\varphi,\theta)| = r^2\cos\theta$.

$$\Rightarrow \int_K f(x,y,z)\,d(x,y,z)$$

$$= \int_{g^{-1}(K)} f(r\cos\varphi\cos\theta,\ r\sin\varphi\cos\theta,\ r\sin\theta)\cdot r^2\cos\theta\ d(r,\varphi,\theta)$$

Aufgabe 3: Polarkoordinaten: $g(r,\varphi) = \begin{pmatrix} r\cdot\cos\varphi \\ r\cdot\sin\varphi \end{pmatrix}$

$$g: [0,R]\times[0,2\pi) \to B_R$$

$$I_R = \int_{B_R} e^{-\frac{x^2+y^2}{2}}\,d(x,y) = \int_0^R\int_0^{2\pi} e^{-\frac{r^2(\cos^2\varphi+\sin^2\varphi)}{2}}\cdot r\,d\varphi\,dr$$

$$= \int_0^R\int_0^{2\pi} e^{-r^2/2}\,r\,d\varphi\,dr = 2\pi\left[-e^{-r^2/2}\right]_0^R = 2\pi\left(1-e^{-R^2/2}\right)$$

$$\lim_{R\to\infty} I_R = 2\pi$$

$$\parallel$$

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,d(x,y).$$

Aufgabe 4: Kugelkoordinaten: $g(r,\varphi,\theta) = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta \\ r\sin\theta \end{pmatrix}$

$$g: [2,3]\times[0,2\pi]\times\left[-\tfrac{\pi}{2},\tfrac{\pi}{2}\right] \longrightarrow K$$

$$\int_K x^2 \, d(x,y,z) = \int_2^3 \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^2\cos^2\varphi\cos^2\theta \cdot r^2\cos\theta \, d\varphi\, d\theta\, dr$$

$$= \underbrace{\left(\int_2^3 r^4\, dr\right)}_{I_1} \underbrace{\left(\int_{-\pi/2}^{\pi/2} \cos^3\theta\, d\theta\right)}_{I_2} \underbrace{\left(\int_0^{2\pi} \cos^2\varphi\, d\varphi\right)}_{I_3}$$

$$I_1 = \frac{1}{5} r^5 \Big|_2^3 = \frac{3^5 - 2^5}{5}$$

$$2 I_3 = \underbrace{\sin\varphi\cos\varphi\Big|_0^{2\pi}}_{=0} + \underbrace{\int_0^{2\pi} \sin^2\varphi\, d\varphi + \int_0^{2\pi}\cos^2\varphi\, d\varphi}_{=2\pi}$$

$$\to I_3 = \pi$$

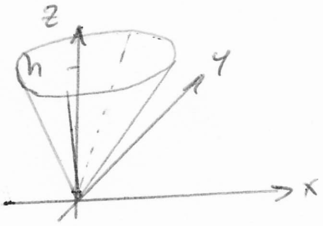
$$3 I_2 = \underbrace{\sin\theta\cos^2\theta\Big|_{-\pi/2}^{\pi/2}}_{=0} + \underbrace{\int_{-\pi/2}^{\pi/2} 2\cos\theta\cdot\sin^2\theta\, d\theta + 2\int_{-\pi/2}^{\pi/2}\cos^3\theta\, d\theta}_{=2\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta}$$

$$= 2\sin\theta\Big|_{-\pi/2}^{\pi/2} = 4$$

$$\to I_2 = 4/3.$$

Beispiel: Integrationsgrenzen

Kegel $K := \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2 \le z^2,\ 0\le z\le h\}$, $h>0$.



$$\int_K \ldots \, d(x,y,z)$$

$$= \int_0^h \int_{-z}^{z} \int_{-\sqrt{z^2+x^2}}^{\sqrt{z^2+x^2}} \ldots \, dy\, dx\, dz$$

$$= \int_0^h \int_{-z}^{z} \int_{-\sqrt{z^2+y^2}}^{\sqrt{z^2+y^2}} \ldots \, dx\, dy\, dz$$

$$= \int_{-h}^{h} \int_{-\sqrt{h^2+x^2}}^{\sqrt{h^2+x^2}} \int_{\sqrt{x^2+y^2}}^{h} \ldots \, dz\, dy\, dx$$

$$= \ldots$$