Linear Algebra II
Tutorial Sheet no. 14

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Exercise T1 (Restriction of bilinear forms)

Consider a bilinear form σ in $\mathbb{R}^n$ and its restriction $\sigma' = \sigma|_U$ to some linear subspace $U \subseteq \mathbb{R}^n$.
Which of the following are generally true? (Give a proof sketch or a counter-example.)

(a) σ symmetric $\Rightarrow \sigma'$ symmetric

(b) σ non-degenerate $\Rightarrow \sigma'$ non-degenerate

(c) σ degenerate $\Rightarrow \sigma'$ degenerate

(d) σ positive definite $\Rightarrow \sigma'$ positive definite

(e) All restrictions σ' for all possible subspaces U are non-degenerate $\Rightarrow \sigma$ either positive definite or negative definite.

Exercise T2 (Matrices over $\mathbb{F}_2$)

(a) Consider the following three matrices $A_i \in \mathbb{F}_2^{(3,3)}$ over the two-element field $\mathbb{F}_2$.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(i) Determine the characteristic polynomials p_{A_i} for $i = 1, 2, 3$ and decompose them into irreducible factors in $\mathbb{F}_2[X]$. List for each of them all eigenvalues together with their geometric multiplicities.

(ii) Which of the matrices A_1, A_2, A_3 are similar to upper triangle matrices over $\mathbb{F}_2$?
Which of them are similar to a Jordan normal form matrix over $\mathbb{F}_2$?
Which of them are diagonalisable over $\mathbb{F}_2$?

(b) (i) Provide precisely one representative for every similarity class of matrices in $\mathbb{F}_2^{(2,2)}$ whose characteristic polynomials split into linear factors.
Hint: consider possible Jordan normal forms.

(ii) Which degree 2 polynomial is irreducible in $\mathbb{F}_2[X]$?
Which matrices in $\mathbb{F}_2^{(2,2)}$ give rise to this characteristic polynomial? Use this to extend the list from (i) to provide precisely one representative for every similarity class of matrices in $\mathbb{F}_2^{(2,2)}$.
Hint: a degree 2 polynomial in $\mathbb{F}_2[X]$ is irreducible iff it has no zeroes over $\mathbb{F}_2$.

Exercise T3 (Polynomials of linear maps)

Let V be a unitary vector space, $\varphi, \psi : V \to V$ endomorphisms of V, and $p, q \in \mathbb{C}[X]$ polynomials. Which of the following statements are always true? Either give a proof or find a counterexample.

(a) If $\varphi \circ \psi = \psi \circ \varphi$, then $p(\varphi) \circ q(\psi) = q(\psi) \circ p(\varphi)$.

(b) Every φ-invariant subspace U of V is also $p(\varphi)$-invariant.

(c) If φ is invertible, then $p(\varphi)$ is also invertible.

(d) If φ is diagonalisable, then $p(\varphi)$ is also diagonalisable.

(e) If φ is unitary, then $p(\varphi)$ is also unitary.

(f) If φ is self-adjoint, then $p(\varphi)$ is also self-adjoint.