Linear Algebra II
Tutorial Sheet no. 12

TECHNISCHE UNIVERSITÄT DARMSTADT

Summer term 2011
June 27, 2011

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise T1 (Warm-up)

Consider a symmetric bilinear form σ in a finite-dimensional euclidean vector space $(V, \langle\cdot,\cdot\rangle)$. Suppose σ has a diagonal representation w.r.t. basis B.

(a) Show that if B consists of pairwise orthogonal basis vectors, then, for every c, the subset

$$\mathbb{X}_c := \{\mathbf{v} \in V : \sigma(\mathbf{v}, \mathbf{v}) = c\}$$

is invariant under reflections in the hyperplanes perpendicular to the basis vectors, $\text{span}(\mathbf{b}_i)^\perp$.

(b) Which property of B or σ guarantees that the sets $\mathbb{X}_c$ also have non-trivial rotational symmetries?

Exercise T2 (Antisymmetric/skew-symmetric bilinear forms)

A bilinear form $\sigma : V \times V \to \mathbb{R}$ is called antisymmetric if $\sigma(\mathbf{v}, \mathbf{w}) = -\sigma(\mathbf{w}, \mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$.

Prove that every bilinear form is the sum of a symmetric bilinear form and an antisymmetric bilinear form, and that decomposition is unique.

Hint (for uniqueness): think of direct sums.

Exercise T3 (Preservation of bilinear forms)

Let σ be a symmetric bilinear form on $\mathbb{R}^n$, represented by $A \in \mathbb{R}^{(n,n)}$ w.r.t. the standard basis. The function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by $Q(\mathbf{x}) = \sigma(\mathbf{x}, \mathbf{x})$ is called the *associated quadratic form* of σ.

We say that an endomorphism φ of $\mathbb{R}^n$ *preserves* the bilinear from σ if $\sigma(\varphi(\mathbf{x}), \varphi(\mathbf{y})) = \sigma(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Analogously, φ preserves the associated quadratic form Q if $Q(\varphi(\mathbf{x})) = Q(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

Show that for an endomorphism φ represented by the matrix C w.r.t. the standard basis, the following are equivalent:

(a) φ preserves Q;

(b) φ preserves σ;

(c) $C^t A C = A$.

Exercise T4 (Diagonalisability of bilinear forms)

Let the bilinear forms σ_1 and σ_2 on $\mathbb{R}^3$ be defined by the matrices

$$A_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the standard basis of $\mathbb{R}^3$.

(a) Is σ_1 or σ_2 degenerate?

(b) Determine for $i = 1, 2$ an orthonormal basis of $\mathbb{R}^3$ with respect to which the matrix of σ_i is diagonal.

(c) What are the signatures of σ_1 and σ_2? Are they positive definite? Is there any plane of symmetry of the "unit surfaces" or any invariance under translation?