Linear Algebra II
Tutorial Sheet no. 10

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Exercise T1 (Warm-up)

Decide whether the following statements are true or false.

(a) Every orthogonal matrix is regular.

(b) Every regular matrix in $\mathbb{R}^{(n,n)}$ is similar to an orthogonal matrix.

(c) $O(n) \subseteq \mathbb{R}^{(n,n)}$ forms a linear subspace.

(d) Orthogonal projections are orthogonal maps.

(e) Permutations matrices are orthogonal.

(f) For the orthogonal projections onto a subspace U and onto its orthogonal complement $U^\perp$ in a finite-dimensional euclidean/unitary space: $\pi_{U^\perp} = \mathrm{id}_V - \pi_U$.

(g) All matrix representations of orthogonal projections of $\mathbb{R}^n$ onto k-dimensional subspaces of the euclidean space $\mathbb{R}^n$ are similar.

(h) All matrix representations of projections of $\mathbb{R}^n$ onto k-dimensional subspaces of $\mathbb{R}^n$ are similar via an orthogonal transformation matrix.

Exercise T2 (Self-adjoint maps and orthogonal projections)

Let $(V, \langle,\rangle)$ be a finite dimensional euclidean or unitary vector space. An endomorphism $\pi : V \to V$ is called *self-adjoint* if $\langle \pi(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{v}, \pi(\mathbf{w})\rangle$ for all $\mathbf{v}, \mathbf{w} \in V$ (cf. Definition 2.4.1 in the notes). Suppose that π is a projection, i.e., $\pi \circ \pi = \pi$. Show that π is self-adjoint if and only if π is an orthogonal projection.

Exercise T3 (Orthogonal maps)

(a) Show that an orthogonal map in $\mathbb{R}^2$ is either the identity, the reflection in the origin, a reflection in a line or a rotation (the first two being special cases of the fourth). Conclude that every orthogonal map in $\mathbb{R}^2$ is the composition of at most two reflections in a line.

(b) Show that an orthogonal map in $\mathbb{R}^3$ is either the identity, a reflection in a plane, a reflection in a line, the reflection in the origin, a rotation about an axis or a rotation about an axis followed by a reflection in the plane orthogonal to the axis (the first four being special cases of the last two). Conclude that every orthogonal map in $\mathbb{R}^3$ is the composition of at most three reflections in a plane.
Extra: how about orthogonal maps in $\mathbb{R}^n$?

[Hint: Take a look at Corollary 2.3.18 in the notes.]

Exercise T4 (Orthogonal maps)

Set

$$A := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Determine an orthogonal matrix P, for which P^tAP is diagonal and compute P^tAP.