Linear Algebra II
Tutorial Sheet no. 9

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Exercise T1 (Warm-up: Normal matrices)

Let $A \in \mathbb{C}^{(n,n)}$ be normal, that is, $AA^+ = A^+A$. Show that if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then it is also an eigenvector of A^+ with eigenvalue $\overline{\lambda}$.

Hint: consider $\langle A^+\mathbf{v} - \overline{\lambda}\mathbf{v},\, A^+\mathbf{v} - \overline{\lambda}\mathbf{v}\rangle$.

Exercise T2 (Matrix groups)

(a) Show that the special orthogonal group $SO(n)$ is the subgroup of $O(n)$ consisting of those matrices that represent orientation preserving orthogonal maps in $\mathbb{R}^n$ w.r.t. the standard orthonormal basis. (Compare Exercise 2.3.11 on page 74 of the notes.)

(b) Prove that $U(1)$ and $SO(2)$ are isomorphic as groups.

[Hint: use that $\mathbb{C}\backslash\{0\}$ is isomorphic to a certain subgroup of $\mathrm{GL}_2(\mathbb{R})$.]

Exercise T3 (Orientation preserving orthogonal maps)

(a) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be an orientation preserving orthogonal map. Show that, for any set of vectors $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^n$, we have

$$\det(\varphi(\mathbf{a}_1), \dots, \varphi(\mathbf{a}_n)) = \det(\mathbf{a}_1, \dots, \mathbf{a}_n)\,.$$

(b) Show that any orientation preserving orthogonal map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ preserves the cross product.

Hint: recall from Exercise (T15.2) from Linear Algebra I that the cross product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is the unique vector $\mathbf{a} \times \mathbf{b} \in \mathbb{R}^3$ such that $\langle \mathbf{a} \times \mathbf{b}, \mathbf{x}\rangle = \det(\mathbf{a}, \mathbf{b}, \mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

(c) Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be the orientation preserving orthogonal map with $\varphi(2,1,2) = (0,3,0)$ and $\varphi(0,-3,0) = (2,-1,-2)$. Determine the matrix representation of φ with respect to the standard basis, and interpret φ geometrically.

Hint: use (b).

Exercise T4 (Orthogonal maps)

(a) Show that an orthogonal map in $\mathbb{R}^2$ is either the identity, the reflection in the origin, a reflection in a line or a rotation (the first two being special cases of the fourth). Conclude that every orthogonal map in $\mathbb{R}^2$ is the composition of at most two reflections in a line.

(b) Show that an orthogonal map in $\mathbb{R}^3$ is either the identity, a reflection in a plane, a reflection in a line, the reflection in the origin, a rotation about an axis or a rotation about an axis followed by a reflection in the plane orthogonal to the axis (the first four being special cases of the last two). Conclude that every orthogonal map in $\mathbb{R}^3$ is the composition of at most three reflections in a plane.

Extra: how about orthogonal maps in $\mathbb{R}^n$?

[Hint: Take a look at Corollary 2.3.18 in the notes.]