Linear Algebra II
Tutorial Sheet no. 8

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Exercise T1 (Warm-up)

Let V be a vector space with basis $B = (b_1, \dots, b_n)$

(a) Give a definition of non-degeneracy for bilinear forms on V, and show that
   i. σ is non-degenerate iff $[\![\sigma]\!]^B$ is regular,
   ii. σ is symmetric/hermitian iff $[\![\sigma]\!]^B$ is symmetric/self-adjoint.

(b) Check for consistency that the change-of-basis transformation for matrices for bilinear forms are such that regularity, symmetry, self-adjointness are preserved.

(c) Let $\approx$ be the "similarity" of real/complex matrices as representations of the same (semi-)bilinear form. Which real/complex $n \times n$ matrices exactly are $\approx$ equivalent to the n-dimensional unit matrix?

Exercise T2 (Orthogonal complements in $\mathbb{R}^3$)

For each of the following subspaces U in $\mathbb{R}^3$, find an orthonormal basis for U, complete this to an orthonormal basis for $\mathbb{R}^3$, and then give an orthonormal basis for $U^\perp$.

(a) $U = \{(x, y, z) \mid x + 2y + 3z = 0\}$.

(b) $U = \{(x, y, z) \mid x + y + z = 0 \text{ and } x - y + z = 0\}$.

Exercise T3 (An orthonormal basis)

Let $V := \mathrm{Pol}_2(\mathbb{R})$ be the $\mathbb{R}$-vector space of all polynomial functions over $\mathbb{R}$ of degree at most 2. On this vector space

$$\langle p_1, p_2 \rangle := \int_{-1}^{1} p_1(x)\, p_2(x)\, \mathrm{d}x$$

defines a scalar product, turning $(V, \langle . , . \rangle)$ into a euclidean space (see Section 2.2 on page 62 of the notes).

Determine an orthonormal basis of V.

Exercise T4 (Dual spaces)

Recall that for any $\mathbb{F}$-vector space V, the set $\mathrm{Hom}(V, \mathbb{F})$ of linear maps $V \to \mathbb{F}$ has again the structure of a vector space, with vector addition and scalar multiplication being defined pointwise, turning it into what is called the dual space of V (see Section 3.2.2 on page 87 of the notes of Linear Algebra I).

If V is a euclidean vector space, we have a map $\varphi_V : V \to \mathrm{Hom}(V, \mathbb{R})$ with $\varphi_V(\mathbf{w}) \in \mathrm{Hom}(V, \mathbb{R})$ for any $\mathbf{w} \in V$ defined by

$$\varphi_V(\mathbf{w})(\mathbf{v}) = \langle \mathbf{w}, \mathbf{v} \rangle, \text{ for all } \mathbf{v} \in V.$$

The aim of this exercise is to show that φ_V is an isomorphism if V is finite dimensional, but not necessarily if V is infinite dimensional.

(a) Show that φ_V is an injective linear map.

(b) Show that φ_V is an isomorphism if V is finite dimensional.

From now on, we consider the sequence space $\mathscr{F}(\mathbb{N}, \mathbb{R})$ and define

$$V = \{f \in \mathscr{F}(\mathbb{N}, \mathbb{R}) : f(n) = 0 \text{ for all but finitely many } n\}.$$

(c) Show that $\langle f, g\rangle = \sum_{n\in\mathbb{N}} f(n)g(n)$ defines a scalar product on the subspace V of $\mathscr{F}(\mathbb{N},\mathbb{R})$, turning $(V, \langle .,.\rangle)$ into a euclidean space. Check that $\langle f, g\rangle$ is defined if $f \in \mathscr{F}(\mathbb{N},\mathbb{R})$ and $g \in V$, but not necessarily if f and g both belong to $\mathscr{F}(\mathbb{N},\mathbb{R})$.

(d) Show that the map $\psi : \mathscr{F}(\mathbb{N},\mathbb{R}) \to \mathrm{Hom}(V,\mathbb{R})$ with $\psi(f) \in \mathrm{Hom}(V,\mathbb{R})$ for any $f \in \mathscr{F}(\mathbb{N},\mathbb{R})$ defined by $\psi(f)(g) = \langle f, g\rangle$ is an isomorphism of vector spaces. Conclude from this that φ_V, which is ψ restricted to V, is not.

Hint: use that the functions $b_n \in V$ $(n \in \mathbb{N})$ defined by $b_n(i) = 1$ if $n = i$ and 0 otherwise, form an orthonormal basis for V.