Linear Algebra II
Tutorial Sheet no. 7

TECHNISCHE UNIVERSITÄT DARMSTADT

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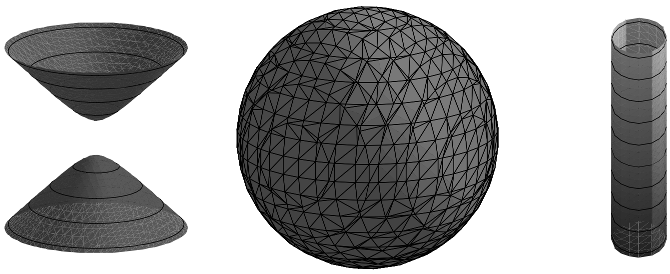
Exercise T1 (Warm up: bilinear form - examples)

Each of the following pictures shows the unit surface $\{\mathbf{v} \in \mathbb{R}^3 : \sigma(\mathbf{v}, \mathbf{v}) = 1\}$ for some bilinear form σ.
Which of these forms is positive definite?
Which are non-degenerate?
For each of these bilinear forms, give an example of a matrix that represents that form.



Exercise T2 (Basis transformations for (semi-)bilinear forms)

Compare Exercise 2.1.3 on page 58 of the notes.

(a) Let σ be a bilinear form on an n-dimensional $\mathbb{R}$-vector space V, represented by the matrix A with respect to the basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$. If $B' = (\mathbf{b}'_1, \dots, \mathbf{b}'_n)$ is another basis for V, find an expression for the matrix $A' = [\![\sigma]\!]^{B'}$ in terms of A, the basis transformation matrices $C = [\![\mathrm{id}_V]\!]^{B}_{B'}$ and $C^{-1} = [\![\mathrm{id}_V]\!]^{B'}_{B}$ as well as their transposes as appropriate.

(b) Similarly for a semi-bilinear form σ of an n-dimensional $\mathbb{C}$-vector space V: if σ is represented by A w.r.t. a basis B, what is its representation A' w.r.t. a basis B' in terms of A, the basis transformations matrices, as well as their adjoints?

(c) Consider the following bilinear form $\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ on the vector space $\mathbb{R}^2$:

$$\sigma(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}) := 7v_1w_1 - 5v_1w_2 - 5v_2w_1 + 4v_2w_2.$$

What is its representation with respect to the standard basis? Then compute its representation with respect to the basis $(\mathbf{b}_1, \mathbf{b}_2) = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix})$ directly, as well as by using the formula obtained in part (a).

(d) Is the bilinear form σ in part (c) symmetric? Is it positive definite?

Exercise T3

Let $\mathscr{F}(\mathbb{N}, \mathbb{C})$ denote the set of sequences $\mathbf{a} = (a_0, a_1, a_2, \dots)$ in $\mathbb{C}$.

(a) Show that given $\mathbf{a}, \mathbf{b}$ in $\mathscr{F}(\mathbb{N}, \mathbb{C})$ for which $\sum_{i=0}^{\infty} |a_i|^2$ and $\sum_{i=0}^{\infty} |b_i|^2$ are convergent, $\sum_{i=0}^{\infty} \bar{a}_i b_i$ is absolutely convergent.

(b) Let $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$ denote the set of sequences $\mathbf{a} \in \mathscr{F}(\mathbb{N}, \mathbb{C})$ for which $\sum_{i=0}^{\infty} |a_i|^2$ converges. Show that $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$ forms a subspace of $\mathscr{F}(\mathbb{N}, \mathbb{C})$, and that the scalar product $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=0}^{\infty} \bar{a}_i b_i$ on $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$ is unitary.

(c) Show that the "generalised standard basis vectors" consisting of sequences with a single 1 and zeroes elsewhere form an infinite family of pairwise orthogonal unit vectors in $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$, but they do not form a basis of $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$.

(d) *** (Cf Exercise H10.3 on the Christmas sheet LA I) Does $\mathscr{F}(\mathbb{N},\mathbb{C})_2$ admit a countable basis?

Exercise T4

In $\mathbb{R}^3$, let ρ be the rotation through an angle of $\frac{\pi}{4}$ about the vector $(1,1,1)$. In this problem we will find the matrix representing ρ w.r.t. the standard basis of $\mathbb{R}^3$.

(a) Find an orthonormal basis $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ of $\mathbb{R}^3$ such that $\mathbf{b}_1$ is a positive scalar multiple of $(1,1,1)$ and $[\![\mathrm{id}]\!]_S^B$ is an orthogonal matrix of determinant 1.

(b) Write down the matrix representing ρ w.r.t. the basis B.

(c) Express the matrix representing ρ using $[\![\rho]\!]_B^B$ and $[\![\mathrm{id}]\!]_S^B$.