Linear Algebra II
Tutorial Sheet no. 6

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Exercise T1 (Warm-up)

Let φ be an endomorphism of an n-dimensional $\mathbb{F}$-vector space V. Assume that φ is represented in the basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ by the matrix in block upper triangle form $M = \begin{pmatrix} A & D \\ 0 & C \end{pmatrix}$ where $A \in \mathbb{F}^{(k,k)}$, $C \in \mathbb{F}^{(n-k,n-k)}$, and $D \in \mathbb{F}^{(k,n-k)}$.

Discuss what this implies about

(a) the existence of invariant subspaces for φ.

(b) the characteristic polynomial of φ.

(c) the minimal polynomial of φ.

Consider examples of various situations of this kind.

Exercise T2 (Jordan normal form)

Write down matrices $A_i \in \mathbb{R}^{(4,4)}$ in Jordan normal form with the following properties:

(a) A_1 has eigenvalues 2 and 4, with 2 having algebraic multiplicity 3 and geometric multiplicity 1.

(b) A_2 has the eigenvalue 5 with algebraic multiplicity 4 and geometric multiplicity 3.

(c) A_3 has the eigenvalue 7 with algebraic multiplicity 2 and geometric multiplicity 2 and the eigenvalue -3 with algebraic multiplicity 2 and geometric multiplicity 1.

(d) The matrices A_4 and A_5 both have the eigenvalue 1 with algebraic multiplicity 4 and geometric multiplicity 2 and have no other eigenvalues. Furthermore, A_4 and A_5 are not similar.

(e) Find two matrices that have the same characteristic and minimal polynomial, yet are not similar.

Exercise T3 (Jordan normal form and transpose)

(a) Show that if the $A, B \in \mathbb{F}^{(n,n)}$ are similar, then so are A^t and B^t.

(b) Let $A \in \mathbb{F}^{(n,n)}$ be a matrix in Jordan normal form. Show that A is similar to A^t. Deduce that over $\mathbb{C}$ every square matrix is similar to its transpose.

Exercise T4 (Square roots)

Consider the set of all 4×4 complex matrices A with characteristic polynomial $p_A(x) = X^4$. We wish to determine exactly which such matrices admit a square root, that is, some matrix S such that $S^2 = A$.

(a) Suppose that A and B are similar matrices. Show that A has a square root if and only if B has a square root. Conclude that it is enough to consider matrices that are in Jordan normal form.

(b) Show that neither of the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

admits a square root.

(c) Show that the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

admit square roots. Conclude that the set of 4×4 matrices with characteristic polynomial X^4 which admit a square root are precisely the ones which are similar to one of these Jordan forms.