Linear Algebra II
Tutorial Sheet no. 5

TECHNISCHE UNIVERSITÄT DARMSTADT

Summer term 2011
May 6, 2011

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise T1 (Polynomials of matrices and linear maps)

In the following p,q stand for polynomials in $\mathbb{F}[X]$, φ for an endomorphism of an n-dimensional $\mathbb{F}$-vector space V, A,B for $n \times n$ matrices over $\mathbb{F}$. Which of these claims are generally true, which are false in general (and which are plain nonsense)?

(a) $p(AB) = p(A)p(B)$ (?)

(b) $(pq)(A) = p(A)q(A) = q(A)p(A)$ (?)

(c) $(p(\varphi))(\mathbf{v}) = p(\varphi(\mathbf{v}))$ (?)

(d) $[\![p(\varphi)]\!]^B_B = p([\![\varphi]\!]^B_B)$ (?)

(e) A regular $\Rightarrow p(A)$ regular (?)

(f) $A \sim B \Rightarrow p(A) \sim p(B)$ (?)

(g) $\varphi(\mathbf{v}) = \lambda\mathbf{v} \Rightarrow (p(\varphi))(\mathbf{v}) = p(\lambda)\mathbf{v}$ (?)

(h) $p(A)q(A) = 0 \Rightarrow (p(A) = 0 \vee q(A) = 0)$ (?)

(i) φ and $p(\varphi)$ have the same invariant subspaces (?)

(j) $U \subseteq V$ an invariant subspace of $\varphi \quad \Rightarrow \quad U$ invariant under $p(\varphi)$ (?)

(k) $U \subseteq V$ an invariant subspace of $\varphi \quad \Rightarrow \quad (p(\varphi))(\mathbf{v}+U) = (p(\varphi))(\mathbf{v}) + U$ (?) (φ viewed as a map on subsets of V.)

(l) $U \subseteq V$ an invariant subspace of $\varphi' \quad \Rightarrow \quad (p(\varphi'))(\mathbf{v}+U) = (p(\varphi))(\mathbf{v}) + U$ (?) (φ' the induced endomorphism of V/U.)

Exercise T2 (Eigenvectors)

Consider the matrices $A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{pmatrix}$ and $B := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}$

(a) Determine the characteristic and minimal polynomials of A and B.

(b) For the matrix B:

i. Show that $\mathbf{v}_1 = (1,0,0,0)$ and $\mathbf{v}_2 = (0,0,1,1)$ are eigenvectors with eigenvalue 2.

ii. Determine an eigenvector $\mathbf{v}_4$ with eigenvalue 3.

iii. Check that $\mathbf{v}_3 = (0,1,0,0)$ is a solution of $(B - 2E_4)^2\mathbf{x} = \mathbf{0}$ and that $B\mathbf{v}_3 = 2\mathbf{v}_3 + \mathbf{v}_1$.

iv. Determine the matrix that represents φ_B w.r.t. the basis $(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_4)$.

Exercise T3 (Complexification)

For $A \in \mathbb{R}^{(2,2)}$ consider the associated endomorphisms $\varphi^{\mathbb{R}}_A$ and $\varphi^{\mathbb{C}}_A$, which are represented by A w.r.t. the standard bases of $\mathbb{R}^2$ and of $\mathbb{C}^2$, respectively.

Let the characteristic polynomial p_A be irreducible in $\mathbb{R}[X]$.

(a) Show that p_A has a pair of complex conjugate zeroes. (Recall that the complex conjugate of $z = \alpha + i\beta$ is $\bar{z} = \alpha - i\beta$.)

(b) Show that $\mathbb{C}^2$ has a basis $B = (\mathbf{v}, \bar{\mathbf{v}})$ of eigenvectors of φ_A consisting of a vector $\mathbf{v}$ with eigenvalue λ, and its complex conjugate $\bar{\mathbf{v}}$, which has eigenvalue $\bar{\lambda}$.

(c) Let $\mathbf{b}_1 = \frac{1}{2}(\mathbf{v} + \bar{\mathbf{v}})$ and $\mathbf{b}_2 = \frac{1}{2i}(\mathbf{v} - \bar{\mathbf{v}})$, which lie in $\mathbb{R}^2$.

i. Show that $B' = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^2$.

ii. Determine the matrix representation of $\varphi_A^{\mathbb{R}}$ w.r.t. basis B' and discuss the similarity of A with a matrix that would suggest the interpretation as "rotation followed by dilation"

Exercise T4 (Simultaneous diagonalisation and polynomials)
Let $A \in \mathbb{R}^{(n,n)}$ be a matrix with n distinct real eigenvalues, and let $B \in \mathbb{R}^{(n,n)}$ be an abitrary matrix such that A and B are simultaneously diagonalisable. Show that there exists a polynomial $p \in \mathbb{R}[X]$ such that $B = p(A)$.

Hint. Recall that, last semester in Linear Algebra I, we have shown in exercise (E14.2) that, given n distinct real numbers $a_1, \dots, a_n \in \mathbb{R}$ and n arbitrary real numbers $b_1, \dots, b_n \in \mathbb{R}$, there exists a polynomial p of degree $n-1$ such that $p(a_i) = b_i$ for all i.