Linear Algebra II
Tutorial Sheet no. 4

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Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise T1 (Algebraic and geometric multiplicity)

Let φ be an endomorphism on a finite dimensional $\mathbb{F}$-vector space V and $\lambda \in \mathbb{F}$ an eigenvalue of φ with geometric multiplicity d and algebraic multiplicity s. Show that $d \leq s$.

Hint: Choose a basis B of V that contains d eigenvectors of φ with eigenvalue λ.

Exercise T2 (Upper triangle shape)

Find a real upper triangular matrix similar to

$$A = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Exercise T3 (Ideals)

Recall that a non-empty subset I of a commutative ring R is called an ideal, if it is closed under addition and under multiplication with arbitrary ring elements. The principal ideal I_a generated by a fixed element $a \in R$ is defined by

$$I_a = \{ra : r \in R\}$$

as the set of all multiples of a (see Definition 1.2.16 on page 22 of the notes).

(a) Verify that I_a is the smallest ($\subseteq$-minimal) ideal containing a.

(b) Let I and J be two ideals in a commutative ring R. Prove that

$$I + J = \{i + j : i \in I, j \in J\}$$

is again an ideal, in fact, the smallest ideal containing both I and J.

(c) Prove that every ideal over $\mathbb{Z}$ is principal. Is the same true in the rings $\mathbb{Z}_n$ ($n \in \mathbb{Z}$)? (As already discussed in H3.3 from LA I 2010/11.)

(d) For two elements $m, n \in \mathbb{Z}$, the set $I_m + I_n$ is an ideal over $\mathbb{Z}$, hence principal. This means that $I_m + I_n = I_k$ for some element $k \in \mathbb{Z}$. Express k in terms of m and n.

(e) For any two ideals I and J in a commutative ring R, find an expression for $I \wedge J$, the largest ideal contained in both I and J. Over the ring $\mathbb{Z}$, how does one determine for any pair $m, n \in \mathbb{Z}$ the $k \in \mathbb{Z}$ such that $I_m \wedge I_n = I_k$?