Linear Algebra II
Tutorial Sheet no. 3

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Exercise T1 (Polynomials and polynomial functions)

Let $\mathbb{F}$ be a field (possibly finite). Recall that $\mathbb{F}[X]$ denotes the ring of polynomials

$$p = \sum_{i=0}^{n} a_i X^i, \quad a_i \in \mathbb{F} \quad n \geq 0.$$

For each element $p = \sum_{i=0}^{n} a_i X^i \in \mathbb{F}[X]$, we obtain a function $\check{p} : \mathbb{F} \to \mathbb{F}$, defined by $\check{p}(\lambda) := \sum_{i=0}^{n} a_i \lambda^i$ for $\lambda \in \mathbb{F}$. Recall that $\mathrm{Pol}(\mathbb{F})$ denotes the set of functions obtained in this way, which we call *polynomial functions*. Let $\check{}: \mathbb{F}[X] \to \mathrm{Pol}(\mathbb{F})$ denote the (surjective) map sending $p \mapsto \check{p}$. (You may refer, *e.g.*, to Exercise 1.2.1. in the lecture notes.)

(a) Show that $\check{}$ is a ring homomorphism. In other words, $\check{0} = 0$ and $\check{1} = 1$, where 0 (resp. 1) may also represent the constant function sending everything to 0 (resp. 1). Also, given $p, q \in \mathbb{F}[X]$, we have $(p+q)\check{} = \check{p} + \check{q}$ and $(pq)\check{} = \check{p}\check{q}$.

(b) Show that $\mathbb{F}[X]$ is infinite-dimensional for any $\mathbb{F}$ by showing that all elements X^k for $k \geq 0$ are linearly independent.

(c) Show that $\mathrm{Pol}(\mathbb{F})$ is finite-dimensional when $\mathbb{F} = \mathbb{F}_p$ for a prime p. Conclude that $\check{}$ is not an isomorphism in this case.

(d) Can you give an upper bound of the dimension of $\mathrm{Pol}(\mathbb{F})$? A better upper bound?

(e) Can you think of any other evaluation maps $\mathbb{F}[X] \to R$ for other rings R, defined in a similar way?

Exercise T2 (The Cayley-Hamilton Theorem for diagonalisable matrices)

(a) Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Recall from Exercise Sheet 2 that the characteristic polynomial $p_A(\lambda) = \lambda^2 - 5\lambda + 4$. Show that the characteristic polynomial of A annihilates A in the sense that $p_A(A) = A^2 - 5A + 4E = 0$, where E denotes the identity matrix and 0 denotes the zero matrix.

(b) Let $D = \mathrm{diag}(d_1, \ldots, d_n)$ be a diagonal matrix. Write down the characteristic polynomial $p_d(\lambda)$. Show that D satisfies its characteristic polynomial in the sense that $p_D(D) = O$ where O denotes the zero $n \times n$ matrix.

(c) Let A be a diagonalisable matrix. Without appeal to the Cayley-Hamilton Theorem (which will later generalise this assertion to all matrices), show that $p_A(A) = O$.

Exercise T3 (Polynomial division with remainder)

Let $p_1(x) = x^6 + 5x^5 + 6x^4 - x^3 + 2x^2 + 5x + 4$ and $p_2(x) = x^2 + 2x$. Using polynomial division with remainder, find the unique polynomials $q(x)$ and $r(x)$ with $\deg(r) < 2 = \deg(p_2)$ such that

$$p_1(x) = q(x)p_2(x) + r(x).$$

(See Definition 1.2.8. and Lemma 1.2.9 in the lecture notes.)