Linear Algebra II
Tutorial Sheet no. 1

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Discuss and compare as many different solution strategies as possible for the following two questions from your exam.

Exercise T1 (Exam problem 2)

Let $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be an ordered basis of an n-dimensional $\mathbb{F}$-vector space V.

(a) Let B' be obtained by replacing $\mathbf{b}_i$ by $\mathbf{b}'_i = \sum_{j=1}^{i} \mathbf{b}_j$ for $1 \leq i \leq n$:

$$B' := (\mathbf{b}_1,\ \mathbf{b}_1 + \mathbf{b}_2,\ \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3,\ \ldots\ , \mathbf{b}_1 + \cdots + \mathbf{b}_n).$$

Determine whether B' always also forms a basis of V.

(b) For $\mathbf{v} \in V$ let

$$B - \mathbf{v} := (\mathbf{b}_1 - \mathbf{v},\ \mathbf{b}_2 - \mathbf{v},\ \ldots\ ,\ \mathbf{b}_n - \mathbf{v}).$$

Show that the set of those $\mathbf{v} \in V$ for which $B - \mathbf{v}$ is *not* a basis of V forms an affine subspace of dimension $n - 1$ (which contains, and is therefore spanned by, the $\mathbf{b}_i$).

Hint: turn the condition that $B - \mathbf{v}$ admits a non-trivial linear combination of $\mathbf{0}$ into a vector equation for $\mathbf{v}$.

Exercise T2 (Exam Problem 4)

In $V = \mathbb{R}^4$, let $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map with

$$\varphi((1,0,0,1)) = (2,0,0,1), \qquad \varphi((2,0,0,1)) = (0,1,1,0),$$
$$\varphi((0,1,1,0)) = (0,1,2,0), \qquad \varphi((0,1,2,0)) = (1,0,0,1).$$

(a) Check that $\mathbf{b}_1 = (1,0,0,1)$, $\mathbf{b}_2 = (2,0,0,1)$, $\mathbf{b}_3 = (0,1,1,0)$, $\mathbf{b}_4 = (0,1,2,0)$ form a basis $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$ of $\mathbb{R}^4$ and determine the matrix representation $[\![\varphi]\!]_B^B$ of φ.
Is φ injective? Does it have an inverse?

(b) Let $S = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ be the standard basis. Derive the matrix representations $[\![\varphi]\!]_S^B$ and $[\![\varphi]\!]_S^S$ from $[\![\varphi]\!]_B^B$ through a systematic application of suitable basis transformation matrices.

Exercise T3 (Complex numbers)

Recall that complex numbers are represented by expressions of the form

$$z = a + bi$$

with $a, b \in \mathbb{R}$, $i \notin \mathbb{R}$ a new constant. Identifying $a \in \mathbb{R}$ with the complex number $a + 0i$ and the new constant i with $0 + 1i$, one may introduce addition and multiplication as the natural extensions of addition and multiplication in $\mathbb{R}$ based on associativity, commutativity, distributivity and the identity $i^2 = -1$. $\mathbb{R}$ thus becomes a subfield of the field of complex numbers.

(a) Let $z_1 = 3 + 4i$ and $z_2 = 5 + 12i$ be complex numbers. Compute

$$z_1^{-1}, \quad z_2^{-1}, \quad z_1^2, \quad z_2^2, \quad \text{and} \quad z_1 z_2,$$

and draw them in the complex plane. Find the complex square roots of i, z_1 and z_2, i.e., solve the equations $x^2 = i, x^2 = z_1, x^2 = z_2$ over $\mathbb{C}$.

(b) Define for $\varphi \in \mathbb{R}$,

$$e^{i\varphi} := \cos\varphi + i\sin\varphi.$$

Show that $e^{i\varphi}e^{i\psi} = e^{i(\varphi+\psi)}$ and $(e^{i\varphi})^n = e^{in\varphi}$ for every natural number n.

(c) Show that every complex number $z \in \mathbb{C}\backslash\{0\}$ can be represented as:

$$z = re^{i\varphi},$$

with $r \in \mathbb{R}_{>0}$. Prove that this representation is unique in the following sense:
$z = se^{i\psi}$ with $s > 0$ implies $r = s$ and $\varphi \equiv \psi \mod 2\pi$.

(d) Use the representation from (c) to

i. give a geometric description of complex multiplication in terms of rotations and rescalings (i.e., dilations or contractions) in $\mathbb{R}^2$.

ii. find all complex solutions of $z^5 = 1$ and draw these in the complex plane. In general, find all solutions to $z^n = w$ for $w \in \mathbb{C}\backslash\{0\}, n \in \mathbb{N}$.