Linear Algebra II
Tutorial Sheet no. 14

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Exercise T1 (Restriction of bilinear forms)

Consider a bilinear form σ in $\mathbb{R}^n$ and its restriction $\sigma' = \sigma|_U$ to some linear subspace $U \subseteq \mathbb{R}^n$.
Which of the following are generally true? (Give a proof sketch or a counter-example.)

(a) σ symmetric $\Rightarrow \sigma'$ symmetric

(b) σ non-degenerate $\Rightarrow \sigma'$ non-degenerate

(c) σ degenerate $\Rightarrow \sigma'$ degenerate

(d) σ positive definite $\Rightarrow \sigma'$ positive definite

(e) All restrictions σ' for all possible subspaces U are non-degenerate $\Rightarrow \sigma$ either positive definite or negative definite.

Solution:

a) is true: $\sigma'(\mathbf{v},\mathbf{w}) = \sigma(\mathbf{v},\mathbf{w}) = \sigma(\mathbf{w},\mathbf{v}) = \sigma'(\mathbf{w},\mathbf{v})$ for all $\mathbf{v},\mathbf{w} \in U$.

b) is false: let σ be the bilinear form corresponding to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and let U be the subspace spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. σ is non-degenerate: For each vector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq 0$ we get $\sigma(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}) = v_1^2 + v_2^2 > 0$. But σ' is degenerate: $\sigma'(\begin{pmatrix} \lambda_1 \\ \lambda_1 \end{pmatrix}, \begin{pmatrix} \lambda_2 \\ \lambda_2 \end{pmatrix}) = \lambda_1\lambda_2 - \lambda_1\lambda_2 = 0$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$.

c) is false. The bilinear form on $\mathbb{R}^2$ represented by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in the standard basis is clearly degenerate, but its restriction to the one-dimensional space spanned by $\mathbf{e}_1$ is non-degenerate.

d) is true: $\sigma'(\mathbf{v},\mathbf{v}) = \sigma(\mathbf{v},\mathbf{v}) \geqslant 0$ and $\sigma'(\mathbf{v},\mathbf{v}) = 0 \Rightarrow \sigma(\mathbf{v},\mathbf{v}) = 0$ and $\sigma(\mathbf{v},\mathbf{v}) = 0 \Rightarrow \mathbf{v} = 0$.

e) is true. If σ is indefinite, there is a non-zero vector $\mathbf{v}$ such that $\sigma(\mathbf{v},\mathbf{v}) = 0$. This implies that the restriction of σ to the subspace spanned by $\mathbf{v}$ is degenerate.

Exercise T2 (Matrices over $\mathbb{F}_2$)

(a) Consider the following three matrices $A_i \in \mathbb{F}_2^{(3,3)}$ over the two-element field $\mathbb{F}_2$.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(i) Determine the characteristic polynomials p_{A_i} for $i = 1,2,3$ and decompose them into irreducible factors in $\mathbb{F}_2[X]$. List for each of them all eigenvalues together with their geometric multiplicities.

(ii) Which of the matrices A_1, A_2, A_3 are similar to upper triangle matrices over $\mathbb{F}_2$?
Which of them are similar to a Jordan normal form matrix over $\mathbb{F}_2$?
Which of them are diagonalisable over $\mathbb{F}_2$?

(b) (i) Provide precisely one representative for every similarity class of matrices in $\mathbb{F}_2^{(2,2)}$ whose characteristic polynomials split into linear factors.
Hint: consider possible Jordan normal forms.
(ii) Which degree 2 polynomial is irreducible in $\mathbb{F}_2[X]$?
Which matrices in $\mathbb{F}_2^{(2,2)}$ give rise to this characteristic polynomial? Use this to extend the list from (i) to provide precisely one representative for every similarity class of matrices in $\mathbb{F}_2^{(2,2)}$.
Hint: a degree 2 polynomial in $\mathbb{F}_2[X]$ is irreducible iff it has no zeroes over $\mathbb{F}_2$.

Solution:

(a) (i) We have $p_{A_1} = x^3 + x^2 + 1$. This has no roots in $\mathbb{F}_2$, so it is irreducible. Therefore A_1 has no eigenvalues. Next, $p_{A_2} = (x+1)(x^2+x+1)$. The factor x^2+x+1 is irreducible since it has no roots in $\mathbb{F}_2$, so the only eigenvalue of A_2 is 1, and it has geometric (as well as algebraic) multiplicity 1. Finally, $p_{A_3} = x(x+1)^2$. A calculation shows that the eigenvalues 0 and 1 both have geometric multiplicity 1.

(ii) A_1 and A_2 are not similar to upper triangular matrices over $\mathbb{F}_2$, since the characteristic polynomials p_{A_1} and p_{A_2} do not split into linear factors. Therefore A_1 and A_2 are neither diagonalizable nor similar to Jordan normal form matrices over $\mathbb{F}_2$.

Since p_{A_3} splits into linear factors, A_3 is similar to an upper triangular matrix over $\mathbb{F}_2$, which is easily seen to be a matrix in Jordan normal form. However, A_3 is not diagonalizable since the eigenvalue 1 only has geometric multiplicity 1.

(b) (i) There are five such similarity classes, with Jordan normal form representatives

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(ii) The polynomial x^2+x+1 is clearly irreducible in $\mathbb{F}_2[X]$, since it has no roots. It is easy to verify that the only two matrices in $\mathbb{F}_2^{(2,2)}$ with this characteristic polynomial are $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, which lie in the same similarity class since $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore adjoining $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ to the list of possible Jordan normal forms above gives us exactly one representative for every similarity class of matrices in $\mathbb{F}_2^{(2,2)}$.

Exercise T3 (Polynomials of linear maps)

Let V be a unitary vector space, $\varphi, \psi : V \to V$ endomorphisms of V, and $p, q \in \mathbb{C}[X]$ polynomials. Which of the following statements are always true? Either give a proof or find a counterexample.

(a) If $\varphi \circ \psi = \psi \circ \varphi$, then $p(\varphi) \circ q(\psi) = q(\psi) \circ p(\varphi)$.
(b) Every φ-invariant subspace U of V is also $p(\varphi)$-invariant.
(c) If φ is invertible, then $p(\varphi)$ is also invertible.
(d) If φ is diagonalisable, then $p(\varphi)$ is also diagonalisable.
(e) If φ is unitary, then $p(\varphi)$ is also unitary.
(f) If φ is self-adjoint, then $p(\varphi)$ is also self-adjoint.

Solution:

a) First, note that $\varphi \circ \psi = \psi \circ \varphi$ implies $\varphi^k \circ \psi^n = \psi^n \circ \varphi^k$, for all $k, n \in \mathbb{N}$. Let $p = \sum_k a_k X^k$ and $q = \sum_k b_k X^k$. Then

$$\begin{aligned} p(\varphi) \circ q(\psi) &= \Big(\sum_k a_k \varphi^k\Big) \circ \Big(\sum_k b_k \psi^k\Big) \\ &= \sum_k \sum_n a_k b_n (\varphi^k \circ \psi^n) \\ &= \sum_k \sum_n a_k b_n (\psi^n \circ \varphi^k) \\ &= \Big(\sum_k b_k \psi^k\Big) \circ \Big(\sum_k a_k \varphi^k\Big) = q(\psi) \circ p(\varphi). \end{aligned}$$

b) Suppose that U is φ-invariant and let $p = \sum_k a_k X^k$. For every $\mathbf{x} \in U$, we have $\varphi(\mathbf{x}) \in U$. It follows that $\varphi^k(\mathbf{x}) \in U$, for every $k \in \mathbb{N}$. Since U is a subspace and, hence, closed under linear combinations, we obtain

$$(p(\varphi))(\mathbf{x}) = \sum_k a_k \varphi^k(\mathbf{x}) \in U\,.$$

c) This is false. For instance, if $\varphi = \mathrm{id}$ and $p = 0$ then φ is invertible, but $p(\varphi) = 0$.

d) Choose a basis B such that

$$[\![\varphi]\!]^B_B = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

is a diagonal matrix. Then

$$[\![p(\varphi)]\!]^B_B = p([\![\varphi]\!]^B_B) = \begin{pmatrix} p(\lambda_1) & & & \\ & p(\lambda_2) & & \\ & & \ddots & \\ & & & p(\lambda_n) \end{pmatrix}$$

is also diagonal.

e) This is false. For instance, if $\varphi = \mathrm{id}$ and $p = 0$ then φ unitary, but $p(\varphi) = 0$, which is clearly not.

f) This claim is false. Suppose that $p = i$ is the constant polynomial with value i. Then $p(\varphi)^+ = (i \cdot \mathrm{id})^+ = -i \cdot \mathrm{id}$ while $p(\varphi^+) = i \cdot \mathrm{id}$.