Linear Algebra II
Tutorial Sheet no. 13

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Summer term 2011
July 5, 2011

Exercise T1 (Warm-up: Determinant revisited)

We consider the real vector space V of symmetric, 2×2 real matrices.

(a) Prove that $\det : V \to \mathbb{R}$ is a a quadratic form.

(b) Determine the matrix of the associated bilinear form with respect to the basis

$$\mathscr{B} = \left(B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

(c) Determine the principal axes and sketch the sets

$$\{\mathbf{v} \in V \mid \det \mathbf{v} = 1\}, \quad \{\mathbf{v} \in V \mid \det \mathbf{v} = -1\}.$$

(as subsets of $\mathbb{R}^3$, when every matrix is identified with its coordinates w.r.t. the basis $\mathscr{B}$).

Solution:

a) Let $A = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_3 \\ b_3 & b_2 \end{pmatrix} \in V$. Then

$$\det(\lambda A) = \lambda^2 a_1 a_2 - \lambda^2 a_3^2 = \lambda^2(a_1 a_2 - a_3^2) = \lambda^2 \det(A).$$

Furthermore

$$\begin{aligned} \sigma_{\det}(A,B) &= \frac{1}{2}\big(\det(A+B) - \det(A) - \det(B)\big) \\ &= \frac{1}{2}\big((a_1+b_1)(a_2+b_2) - (a_3+b_3)^2\big) - (a_1 a_2 - a_3^2) - (b_1 b_2 - b_3^2) \\ &= \frac{1}{2}(a_1 b_2 + a_2 b_1 - 2a_3 b_3). \end{aligned}$$

Let $C = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} \in V$ and $\lambda, \mu \in \mathbb{R}$. It follows that

$$\begin{aligned} \sigma_{\det}(\lambda A + \mu C, B) &= \frac{1}{2}\big((\lambda a_1 + \mu c_1) b_2 + (\lambda a_2 + \mu c_2) b_1 - 2(\lambda a_3 + \mu c_3) b_3\big) \\ &= \frac{1}{2}\lambda(a_1 b_2 + a_2 b_1 - 2a_3 b_3) + \frac{1}{2}\mu(c_1 b_2 + c_2 b_1 - 2c_3 b_3) \\ &= \lambda \sigma_{\det}(A,B) + \mu \sigma_{\det}(C,B), \end{aligned}$$

hence $\sigma_{\det}$ is a bilinear form.

b) We have to compute $\sigma_{\det}(B_i, B_j)$ for $i,j = 1,2,3$. We get $\sigma_{\det}(B_1,B_1) = \sigma_{\det}(B_2,B_2) = 0$, $\sigma_{\det}(B_3,B_3) = -1$, $\sigma_{\det}(B_1,B_2) = \sigma_{\det}(B_2,B_1) = \frac{1}{2}$, $\sigma_{\det}(B_1,B_3) = \sigma_{\det}(B_3,B_1) = 0$ und $\sigma_{\det}(B_2,B_3) = \sigma_{\det}(B_3,B_1) = 0$. Therefore we get the following matrix:

$$M = [\![\sigma_{\det}]\!]^B = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

c) We apply principal axes transformation to M. For this we determine first the eigenvalues and eigenvectors of M.

$$p_M = \det(M - \lambda E) = -(\lambda + 1)\left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right).$$

The eigenvalues of M are then $\lambda_1 = -1, \lambda_2 = \frac{1}{2}, \lambda_3 = -\frac{1}{2}$. The corresponding normalized eigenvectors are

$$\mathscr{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathscr{V}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathscr{V}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Translated back to matrices, we get that the three principal axes are the three one-dimensional subspaces of V spanned by the matrices

$$M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By identifying every matrix with its coordinates w.r.t. the basis $\mathscr{B}$, the subspaces become subsets of $\mathbb{R}^3$. Hence, the quadric $\{\mathbf{v} \in V \mid \det \mathbf{v} = 1\}$ describes a two-sheet hyperboloid with the principal axes generated by the vectors $\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3 \in \mathbb{R}^3$. The set $\{\mathbf{v} \in V \mid \det \mathbf{v} = -1\}$ describes a single-sheet hyperboloid.

Exercise T2 (A quadric up to rotation/translation)

Consider the quadratic $\mathbb{X}$ given by $3x^2 + 3y^2 - 2xy + 20x - 12y + 40 = 0$. Our goal is to find the principal axes and describe the graph of $\mathbb{X}$.

(a) Regarding the quadratic part of the above equation as a quadratic form, diagonalise the associated symmetric bilinear form to obtain a basis for which the cross term xy vanishes.

(b) Working in this new basis, eliminate the linear terms by a translation.

(c) Describe $\mathbb{X}$.

Solution:

a) The quadratic form is $Q(x) = 3x^2 - 2xy + 3y^2$, so the corresponding symmetric bilinear form is represented with respect to the standard basis by the matrix $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. The eigenvalues of A and 2 and 4, and the corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, respectively. Letting C be the orthogonal matrix with columns $(\mathbf{v}_1, \mathbf{v}_2)$ we see that $C^tAC = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

b) Let $u = \frac{1}{2}(x + y)$ and $v = \frac{1}{2}(-x + y)$. Then $y = u + v$ and $x = u - v$, and the equation for $\mathbb{X}$ can be rewritten as

$$4u^2 + 8v^2 + 8u - 32v + 40.$$

A translation yields $4(u+1)^2 + 8(v-2)^2 = 4$

c) $\mathbb{X}$ is in an ellipse centred at the point $(-1, 2)$ that has been rotated by $\frac{\pi}{4}$. Its principal axes are $\operatorname{span}(1, -1)$ and $\operatorname{span}(1, 1)$, respectively with major and minor diameters 2 and $\sqrt{2}$.

Exercise T3 (Slicing a quadric)

Consider the quadric $\mathbb{X}_{\lambda,\mu}$ in $\mathbb{R}^3$ defined by

$$\mathbb{X}_{\lambda,\mu} := \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \lambda(x_1^2 + x_2^2) + \mu x_3^2 = 1\},$$

where λ and μ are real parameters.

(a) Determine the intersection of every $\mathbb{X}_{\lambda,\mu}$ with the plane defined by $x_3 = c \in \mathbb{R}$.

(b) Prove that $\mathbb{X}_{\lambda,\mu}$ can be obtained by rotating the set

$$\mathbb{X}'_{\lambda,\mu} := \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0,\ \lambda x_2^2 + \mu x_3^2 = 1\}$$

about the x_3-axis.

(c) For each pair of values

	λ	μ
1.	-1	1
2.	1	-1
3.	2	1

,

sketch $\mathbb{X}_{\lambda,\mu}$ and $\mathbb{X}'_{\lambda,\mu}$.

Solution:

a) The intersection of $\mathbb{X}_{\lambda,\mu}$ with the plane $x_3 = c$ is given by

$$\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \lambda(x_1^2 + x_2^2) = 1 - \mu c^2,\ x_3 = c\}.$$

In case $\frac{1-\mu c^2}{\lambda} \geqslant 0$, this is a circle with radius $r = \sqrt{\frac{1-\mu c^2}{\lambda}}$, otherwise the intersection is empty.

b) Rotating about the x_3-axis does not change the x_3-coordinate of a point. So we look at the planes defined by $x_3 = c$. The points obtained from a point $\mathbf{p} = (0, b, c) \in \mathbb{X}'_{\lambda,\mu}$ by rotation about the x_3-axis are the ones that have the same distance to the x_3-axis as $\mathbf{p}$, hence the points (x_1, x_2, c) satisfying $x_1^2 + x_2^2 = b^2$. From $\lambda b^2 + \mu c^2 = 1$, we obtain $\lambda(x_1^2 + x_2^2) + \mu c^2 = 1$, so all these points are in $\mathbb{X}_{\lambda,\mu}$.

Conversely, every point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{X}_{\lambda,\mu}$ can be obtained by rotating a point $\mathbf{p} = (0, b, c)$ with $b = \pm\sqrt{x_1^2 + x_2^2}$ and $c = x_3$ through an appropriate angle.

c) In the first case, $\mathbb{X}'_{-1,1}$ is a north-south opening hyperbola in the $x_2{-}x_3$-plane, which consists of $x_3 = +\sqrt{1 + x_2^2}$ and $x_3 = -\sqrt{1 + x_2^2}$. Hence $\mathbb{X}_{-1,1}$ is a two-sheet hyperboloid.

In the second case, $\mathbb{X}'_{1,-1}$ is an east-west opening hyperbola in the $x_2{-}x_3$-plane, which consists of $x_2 = +\sqrt{1 + x_3^2}$ and $x_2 = -\sqrt{1 + x_3^2}$; so it is the previous figure, but tilted. Therefore $\mathbb{X}_{1,-1}$ is a single-sheet hyperboloid.

In the third case, $\mathbb{X}'_{2,1}$ is an ellipse in the $x_2{-}x_3$-plane, which consists of $x_3 = +\sqrt{1 - 2x_2^2}$ and $x_3 = -\sqrt{1 - 2x_2^2}$. $\mathbb{X}_{2,1}$ is therefore an ellipsoid.