Linear Algebra II
Tutorial Sheet no. 12

TECHNISCHE UNIVERSITÄT DARMSTADT

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Summer term 2011
June 27, 2011

Exercise T1 (Warm-up)

Consider a symmetric bilinear form σ in a finite-dimensional euclidean vector space $(V, \langle\cdot,\cdot\rangle)$. Suppose σ has a diagonal representation w.r.t. basis B.

(a) Show that if B consists of pairwise orthogonal basis vectors, then, for every c, the subset

$$\mathbb{X}_c := \{\mathbf{v} \in V : \sigma(\mathbf{v}, \mathbf{v}) = c\}$$

is invariant under reflections in the hyperplanes perpendicular to the basis vectors, $\text{span}(\mathbf{b}_i)^\perp$.

(b) Which property of B or σ guarantees that the sets $\mathbb{X}_c$ also have non-trivial rotational symmetries?

Solution:

a) Since σ has a diagonal representation w.r.t. basis B, for $i \neq j$ we have $\sigma(v_j, v_i) = 0$ for all $v_j \in \text{span}(\mathbf{b}_j)$ and $v_i \in \text{span}(\mathbf{b}_i)$. Now let $v = v_{-i} + v_i$ where $v_{-i} \in \text{span}(\mathbf{b}_i)^\perp$ and $\mathbf{v}_i \in \text{span}(\mathbf{b}_i)$. Then $\sigma(\mathbf{v}_{-i} - \mathbf{v}_i, \mathbf{v}_{-i} - \mathbf{v}_i) = \sigma(\mathbf{v}_{-i}, \mathbf{v}_{-i}) + \sigma(\mathbf{v}_i, \mathbf{v}_i) = \sigma(\mathbf{v}_{-i} + \mathbf{v}_i, \mathbf{v}_{-i} + \mathbf{v}_i)$ thanks to the remark above.

b) Assume that B is an orthonormal basis. Let λ be a scalar and let I be such that $\sigma(\mathbf{b}_i, \mathbf{b}_i) = \lambda$ for all $i \in I$. Let $\varphi : V \to V$ be such that $i \notin I \Rightarrow \varphi(\mathbf{b}_i) = \mathbf{b}_i$ and such that $\varphi(\text{span}\{\mathbf{b}_i | i \in I\}) \subseteq \text{span}\{\mathbf{b}_i | i \in I\}$ and the restriction of φ to $\text{span}\{\mathbf{b}_i | i \in I\}$ is isometric. On $\text{span}\{\mathbf{b}_i | i \in I\}$, σ is a scalar multiple of the standard scalar product, so φ preserves σ on $\text{span}\{\mathbf{b}_i | i \in I\}$. Now let $v_I \in \text{span}\{\mathbf{b}_i | i \in I\}$ and $v_{-I} \in \text{span}\{\mathbf{b}_i | i \notin I\}$. We have $\sigma(\varphi(\mathbf{v}_{-I}+\mathbf{v}_I), \varphi(\mathbf{v}_{-I}+\mathbf{v}_I)) = \sigma(\mathbf{v}_{-I}+\varphi(\mathbf{v}_I), \mathbf{v}_{-I}+\varphi(\mathbf{v}_I)) = \sigma(\mathbf{v}_{-I}, \mathbf{v}_{-I}) + \sigma(\varphi(\mathbf{v}_I), \varphi(\mathbf{v}_I)) = \sigma(\mathbf{v}_{-I}, \mathbf{v}_{-I}) + \sigma(\mathbf{v}_I, \mathbf{v}_I) = \sigma(\mathbf{v}_{-I} + \mathbf{v}_I, \mathbf{v}_{-I} + \mathbf{v}_I)$.

Exercise T2 (Antisymmetric/skew-symmetric bilinear forms)

A bilinear form $\sigma : V \times V \to \mathbb{R}$ is called antisymmetric if $\sigma(\mathbf{v}, \mathbf{w}) = -\sigma(\mathbf{w}, \mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$.

Prove that every bilinear form is the sum of a symmetric bilinear form and an antisymmetric bilinear form, and that decomposition is unique.

Hint (for uniqueness): think of direct sums.

Solution:

If $\sigma : V \times V \to \mathbb{R}$ is a bilinear form, then we define by

$$\sigma_s(\mathbf{v}, \mathbf{w}) := \frac{1}{2}\big(\sigma(\mathbf{v}, \mathbf{w}) + \sigma(\mathbf{w}, \mathbf{v})\big) \quad \text{and} \quad \sigma_a(\mathbf{v}, \mathbf{w}) := \frac{1}{2}\big(\sigma(\mathbf{v}, \mathbf{w}) - \sigma(\mathbf{w}, \mathbf{v})\big)$$

two bilinear forms $\sigma_s, \sigma_a : V \times V \to \mathbb{R}$. By construction, σ_s is symmetric and σ_a is antisymmetric. Furthermore, $\sigma = \sigma_s + \sigma_a$.

For uniqueness, let $\sigma_s + \sigma_a = \sigma'_s + \sigma'_a$ where σ_s and σ'_s are symmetric and σ_a and σ'_a are antisymmetric. Then $\sigma_s - \sigma'_s = \sigma_a - \sigma'_a$ is both symmetric and antisymmetric, and is therefore the null form.

Exercise T3 (Preservation of bilinear forms)

Let σ be a symmetric bilinear form on $\mathbb{R}^n$, represented by $A \in \mathbb{R}^{(n,n)}$ w.r.t. the standard basis. The function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by $Q(\mathbf{x}) = \sigma(\mathbf{x}, \mathbf{x})$ is called the *associated quadratic form* of σ.

We say that an endomorphism φ of $\mathbb{R}^n$ *preserves* the bilinear from σ if $\sigma(\varphi(\mathbf{x}), \varphi(\mathbf{y})) = \sigma(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Analogously, φ preserves the associated quadratic form Q if $Q(\varphi(\mathbf{x})) = Q(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

Show that for an endomorphism φ represented by the matrix C w.r.t. the standard basis, the following are equivalent:

(a) φ preserves Q;

(b) φ preserves σ;

(c) $C^t AC = A$.

Solution:

a) $\Rightarrow$ (b) Since

$$\sigma(\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}) = \sigma(\mathbf{x},\mathbf{x}) + 2\sigma(\mathbf{x},\mathbf{y}) + \sigma(\mathbf{y},\mathbf{y})$$

we have

$$\sigma(\mathbf{x},\mathbf{y}) = \frac{1}{2}[Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})].$$

Hence, if φ preserves Q, then it also preserves σ.

b) $\Rightarrow$ (c) If φ preserves σ, then

$$\mathbf{x}^t A\mathbf{y} = \sigma(\mathbf{x},\mathbf{y}) = \sigma(\varphi(\mathbf{x}),\varphi(\mathbf{y})) = \mathbf{x}^t C^t AC\mathbf{y} \quad \text{for all } \mathbf{x},\mathbf{y} \in \mathbb{R}^n.$$

It follows that $A = C^t AC$. (In order to extract the entries in position (i,j) look at the i-th and j-th standard basis vector for $\mathbf{x}$ and $\mathbf{y}$, respectively.)

c) $\Rightarrow$ (a) If $A = C^t AC$, then $Q(\varphi(\mathbf{x})) = \mathbf{x}^t C^t AC\mathbf{x} = \mathbf{x}^t A\mathbf{x} = Q(\mathbf{x})$.

Exercise T4 (Diagonalisability of bilinear forms)
Let the bilinear forms σ_1 and σ_2 on $\mathbb{R}^3$ be defined by the matrices

$$A_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the standard basis of $\mathbb{R}^3$.

(a) Is σ_1 or σ_2 degenerate?

(b) Determine for $i = 1,2$ an orthonormal basis of $\mathbb{R}^3$ with respect to which the matrix of σ_i is diagonal.

(c) What are the signatures of σ_1 and σ_2? Are they positive definite? Is there any plane of symmetry of the "unit surfaces" or any invariance under translation?

Solution:

a) σ_1 is non degenerate since A_1 has rank 3, and σ_2 is degenerate since A_2 has rank 2.

b) We determine an orthonormal basis B_i of eigenvectors for each of the matrices A_i. These eigenvectors form the columns of a transformation matrix C_i. Then $[\![\sigma_i]\!]^{B_i} = {C_i}^t A_i C_i$ is the matrix of σ_i in this basis and, by Proposition 3.2.9 (A_i is symmetric), $[\![\sigma_i]\!]^{B_i}$ is diagonal.

For σ_1: The eigenvalues of A_1 are given by

$$0 = \det(A_1 - \lambda E) = -(3-\lambda)(2-\lambda)(2+\lambda),$$

hence they are 3, 2, and -2. We determine now the eigenvectors:

For $\lambda = 3$:

$$\ker(A_1 - 3E) = \operatorname{span}(\mathbf{v}_1), \quad \text{where } \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda = 2$:

$$\ker(A_1 - E) = \operatorname{span}(\mathbf{v}_2), \quad \text{where } \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

For $\lambda = -2$:

$$\ker(A_1 + E) = \operatorname{span}(\mathbf{v}_3), \quad \text{where } \mathbf{v}_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then $B_1 = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $[\![\sigma_1]\!]^{B_1} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

For σ_2: The eigenvalues of A_1 are given by

$$0 = \det(A_1 - \lambda E) = \lambda(2 - \lambda^2),$$

hence they are 0, $\sqrt{2}$, and $-\sqrt{2}$. We determine now the eigenvectors:
For $\lambda = 0$:

$$\ker(A_1) = \operatorname{span}(\mathbf{v}_1), \quad \text{where } \mathbf{v}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

For $\lambda = \sqrt{2}$:

$$\ker(A_1 - \sqrt{2}E) = \operatorname{span}(\mathbf{v}_2), \quad \text{where } \mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix}.$$

For $\lambda = -\sqrt{2}$:

$$\ker(A_1 + \sqrt{2}E) = \operatorname{span}(\mathbf{v}_3), \quad \text{where } \mathbf{v}_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Then $B_2 = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $[\![\sigma_2]\!]^{B_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$.

c) The signatures are $(+,+,-)$ and $(+,-,0)$ respectively. Neither σ_1 nor σ_2 is positive definite. For σ_1 and σ_2, each $\operatorname{span}(\mathbf{v}_i)^\perp$ is a plane of symmetry of the "unit surfaces". Moreover the "unit surface" of σ_2 is invariant by translation along the corresponding $\operatorname{span}(\mathbf{v}_1)$.