Linear Algebra II
Tutorial Sheet no. 11

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Exercise T1 (Warmup: Skew-hermitian and skew-symmetric matrices)

A matrix $A \in \mathbb{C}^{(n,n)}$ is called skew-hermitian if $A^+ = -A$. Similarly, in the real case, $A \in \mathbb{R}^{(n,n)}$ is called skew-symmetric if $A = -A^t$.

(a) Show that any skew-hermitian or skew-symmetric matrix is normal.

(b) Conclude that for any skew-hermitian matrix A, there exists a unitary matrix U such that $UAU^{-1} = D$, where D is diagonal.

(c) Let $A \in \mathbb{C}^{(n,n)}$ be skew-hermitian. What can you say about the eigenvalues of A?

Solution:

a) If $A^+ = -A$ then $AA^+ = -A^2 = A^+A$.

b) By Corollary 2.4.11 in the notes.

c) $A^+ = (UDU^+)^+ = UD^+U^+$ on the one hand, and $A^+ = -A = -UDU^+$ on the other hand, so $D = -D^+$. Therefore the eigenvalues are pure imaginary.

Exercise T2 (Self-adjoint and normal endomorphisms)

Let V be a finite dimensional euclidean or unitary space and φ an endomorphism of V. Prove the following.

(a) If V is euclidean, then

$$\varphi \text{ is self-adjoint} \iff V \text{ has an orthonormal basis consisting of eigenvectors of } \varphi.$$

(b) If V is unitary, which one of the implications from (a) does not hold?

(c) If V is unitary, then

$$\varphi \text{ is normal} \iff V \text{ has an orthonormal basis consisting of eigenvectors of } \varphi.$$

Solution:

a) $\Rightarrow$ is Proposition 2.4.5. $\Leftarrow$ Let B be an orthonormal basis of V consisting of eigenvectors of φ. The matrix of φ with respect to this basis is diagonal. Since any diagonal matrix is symmetric, it follows that φ is self-adjoint.

b) $\Rightarrow$ holds in a unitary space, by Proposition 2.4.5. The converse does not hold: take $\varphi = i \cdot id_V$. Then V has an orthonormal basis consisting of eigenvectors of φ, since every vector is an eigenvector of this map, so any orthonormal basis of V will do. On the other hand, φ is not self-adjoint, since its adjoint is $\varphi^+ := -i \cdot id_V$.

c) $\Rightarrow$ is Theorem 2.4.10.

$\Leftarrow$ Let B be an orthonormal basis consisting of eigenvectors of φ. As $[\![\varphi]\!]^B_B$ is diagonal, $[\![\varphi^+]\!]^B_B = ([\![\varphi]\!]^B_B)^+$ is diagonal, too. Since any two diagonal matrices commute, so do φ and φ^+.

Exercise T3 (Orthogonal diagonalisability)

Find an *orthogonal* matrix C such that the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

is transformed into a diagonal matrix by $C^{-1}AC = C^tAC$. Which property of A guarantees that you can find such a C?

[Hint: The charactaristic polynomial is $p_A = (X-1)^2(X-4)$]

Solution:

The matrix A is real symmetric, and therefore is similar to a diagonal matrix using an orthogonal transformation matrix C (Corollary 2.4.6 on page 77 of the notes).

To compute C, we first note that the characteristic polynomial is $p_A = (X-1)^2(X-4)$. Thus the eigenvalues of A are 1 (with multiplicity 2) and 4.

For the eigenvalue $\lambda = 4$, we get the eigenspace: $\ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \operatorname{span}\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \}$.

For the eigenvalue $\lambda = 1$, we get the eigenspace: $\ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \operatorname{span}\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \}$.

Since we look for an orthogonal transformation matrix C, we have to use Gram-Schmidt on the latter eigenspace:

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \tfrac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

After normalising the vectors, we obtain the following matrix C:

$$C = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix},$$

and $C^tAC = C^{-1}AC = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.