Linear Algebra II
Tutorial Sheet no. 10

TECHNISCHE UNIVERSITÄT DARMSTADT

Summer term 2011
June 14, 2011

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Exercise T1 (Warm-up)

Decide whether the following statements are true or false.

(a) Every orthogonal matrix is regular.

(b) Every regular matrix in $\mathbb{R}^{(n,n)}$ is similar to an orthogonal matrix.

(c) $O(n) \subseteq \mathbb{R}^{(n,n)}$ forms a linear subspace.

(d) Orthogonal projections are orthogonal maps.

(e) Permutations matrices are orthogonal.

(f) For the orthogonal projections onto a subspace U and onto its orthogonal complement $U^\perp$ in a finite-dimensional euclidean/unitary space: $\pi_{U^\perp} = \mathrm{id}_V - \pi_U$.

(g) All matrix representations of orthogonal projections of $\mathbb{R}^n$ onto k-dimensional subspaces of the euclidean space $\mathbb{R}^n$ are similar.

(h) All matrix representations of projections of $\mathbb{R}^n$ onto k-dimensional subspaces of $\mathbb{R}^n$ are similar via an orthogonal transformation matrix.

Solution:

a) True, since its determinant is ± 1.

b) False. The determinant is invariant under similarity transformations, but not every regular matrix has determinant ± 1.

c) False. The null matrix is not orthogonal!

d) False, since (nontrivial) orthogonal projections are not regular.

e) True, since the set of columns of an $n \times n$ permutation matrix is precisely the standard basis for $\mathbb{R}^n$, suitably permuted.

f) True. This is clear from the definition of orthogonal projection.

g) True. Every (orthogonal) projection of rank k is represented by a diagonal matrix with k ones and $n-k$ zeroes on the diagonal (w.r.t. any basis obtained from bases of the image and the kernel, which in case of an orthogonal projection can even be chosen to form an onb).

h) False. A non-orthogonal projection π, for which $\mathrm{image}(\pi) \neq (\ker(\pi))^\perp$ cannot be isometrically related to an orthogonal projection.

Exercise T2 (Self-adjoint maps and orthogonal projections)

Let $(V, \langle , \rangle)$ be a finite dimensional euclidean or unitary vector space. An endomorphism $\pi : V \to V$ is called *self-adjoint* if $\langle \pi(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, \pi(\mathbf{w}) \rangle$ for all $\mathbf{v}, \mathbf{w} \in V$ (cf. Definition 2.4.1 in the notes). Suppose that π is a projection, i.e., $\pi \circ \pi = \pi$. Show that π is self-adjoint if and only if π is an orthogonal projection.

Solution:

(⇒) Assume that π is self-adjoint. Let $\mathbf{v} \in \ker(\pi)$ and $\mathbf{w} \in \mathrm{Im}(\pi)$. Then $\pi(\mathbf{w}) = \mathbf{w}$. Hence,

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \pi(\mathbf{w}) \rangle = \langle \pi(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{0}, \mathbf{w} \rangle = 0.$$

Therefore $\ker(\pi) \perp \mathrm{Im}(\pi)$ and π is an orthogonal projection, by exercise (E8.4)

(⇐) When π is an orthogonal projection, then it is the orthogonal projection onto $U := \mathrm{Im}(\pi)$. Decomposing $\mathbf{v}, \mathbf{w} \in V$ as $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ and $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1$ where $\mathbf{v}_0, \mathbf{w}_0 \in U$, $\mathbf{v}_1, \mathbf{w}_1 \in U^\perp$, we have $\pi(\mathbf{v}) = \mathbf{v}_0$ and $\pi(\mathbf{w}) = \mathbf{w}_0$. Therefore

$$\langle \pi(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}_0, \mathbf{w}_0 + \mathbf{w}_1 \rangle = \langle \mathbf{v}_0, \mathbf{w}_0 \rangle = \langle \mathbf{v}_0 + \mathbf{v}_1, \mathbf{w}_0 \rangle = \langle \mathbf{v}, \pi(\mathbf{w}) \rangle .$$

Exercise T3 (Orthogonal maps)

(a) Show that an orthogonal map in $\mathbb{R}^2$ is either the identity, the reflection in the origin, a reflection in a line or a rotation (the first two being special cases of the fourth). Conclude that every orthogonal map in $\mathbb{R}^2$ is the composition of at most two reflections in a line.

(b) Show that an orthogonal map in $\mathbb{R}^3$ is either the identity, a reflection in a plane, a reflection in a line, the reflection in the origin, a rotation about an axis or a rotation about an axis followed by a reflection in the plane orthogonal to the axis (the first four being special cases of the last two). Conclude that every orthogonal map in $\mathbb{R}^3$ is the composition of at most three reflections in a plane.
Extra: how about orthogonal maps in $\mathbb{R}^n$?

[Hint: Take a look at Corollary 2.3.18 in the notes.]

Solution:

a) Corollary 2.3.18 gives the following matrix representations for an orthogonal map of $\mathbb{R}^2$ w.r.t. a suitably chosen orthonormal basis:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

So it is either the identity, the reflection in the origin, a reflection in a line or a rotation. Every rotation is the composition of two reflections in a line (the rotation through an angle α is the composition of the reflection in x-axis, followed by the reflection in the line that makes an angle $\frac{1}{2}\alpha$ with the x-axis). This implies that every orthogonal map in $\mathbb{R}^2$ is the composition of at most two reflections in a line.

b) In the same manner, we obtain the following matrix representations for an orthogonal map of $\mathbb{R}^3$ w.r.t. a suitably chosen orthonormal basis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}.$$

So an orthogonal map of $\mathbb{R}^3$ is either the identity, a reflection in a plane, a reflection in a line, the reflection in the origin, a rotation about an axis or a rotation about an axis followed by a reflection in the plane orthogonal to the axis.

We see that every orthogonal map in $\mathbb{R}^3$ is the composition of at most three reflections in a plane. In general, any orthogonal map in $\mathbb{R}^n$ is the composition of at most n reflections in $(n-1)$-dimensional hyperplanes.

Exercise T4 (Orthogonal maps)

Set

$$A := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Determine an orthogonal matrix P, for which $P^t A P$ is diagonal and compute $P^t A P$.

Solution:

The characteristic polynomial is $p_A = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$, so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. As corresponding eigenvectors we find $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for λ_1 and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for λ_2. They are already orthogonal, so we only need to normalise them to obtain the onb $B = (\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) = (\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix})$. Then, writing S for the standard basis $S = (\mathbf{e}_1, \mathbf{e}_2)$, we get

$$P = [\![\mathrm{id}]\!]_S^B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \; P^t = P^{-1} = [\![\mathrm{id}]\!]_B^S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \; P^t A P = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$