Linear Algebra II
Tutorial Sheet no. 9

Summer term 2011
June 6, 2011

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise T1 (Warm-up: Normal matrices)

Let $A \in \mathbb{C}^{(n,n)}$ be normal, that is, $AA^+ = A^+A$. Show that if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then it is also an eigenvector of A^+ with eigenvalue $\overline{\lambda}$.

Hint: consider $\langle A^+\mathbf{v} - \overline{\lambda}\mathbf{v},\ A^+\mathbf{v} - \overline{\lambda}\mathbf{v}\rangle$.

Solution:

Assume that $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$\begin{aligned}
\langle A^+\mathbf{v} - \overline{\lambda}\mathbf{v}, A^+\mathbf{v} - \overline{\lambda}\mathbf{v}\rangle &= \langle A^+\mathbf{v}, A^+\mathbf{v}\rangle - \langle A^+\mathbf{v}, \overline{\lambda}\mathbf{v}\rangle - \langle \overline{\lambda}\mathbf{v}, A^+\mathbf{v}\rangle + \langle \overline{\lambda}\mathbf{v}, \overline{\lambda}\mathbf{v}\rangle \\
&= \langle \mathbf{v}, AA^+\mathbf{v}\rangle - \overline{\lambda}\langle \mathbf{v}, A\mathbf{v}\rangle - \lambda\langle \mathbf{v}, A^+\mathbf{v}\rangle + \overline{\lambda}\lambda\langle \mathbf{v}, \mathbf{v}\rangle \\
&= \langle \mathbf{v}, A^+A\mathbf{v}\rangle - \overline{\lambda}\langle \mathbf{v}, A\mathbf{v}\rangle - \lambda\langle A\mathbf{v}, \mathbf{v}\rangle + \overline{\lambda}\lambda\langle \mathbf{v}, \mathbf{v}\rangle \\
&= \lambda\langle \mathbf{v}, A^+\mathbf{v}\rangle - \overline{\lambda}\lambda\langle \mathbf{v}, \mathbf{v}\rangle - \lambda\overline{\lambda}\langle \mathbf{v}, \mathbf{v}\rangle + \overline{\lambda}\lambda\langle \mathbf{v}, \mathbf{v}\rangle \\
&= \lambda\langle A\mathbf{v}, \mathbf{v}\rangle - \overline{\lambda}\lambda\langle \mathbf{v}, \mathbf{v}\rangle \\
&= 0\,.
\end{aligned}$$

Hence, $A^+\mathbf{v} - \overline{\lambda}\mathbf{v} = \mathbf{0}$.

Exercise T2 (Matrix groups)

(a) Show that the special orthogonal group $SO(n)$ is the subgroup of $O(n)$ consisting of those matrices that represent orientation preserving orthogonal maps in $\mathbb{R}^n$ w.r.t. the standard orthonormal basis. (Compare Exercise 2.3.11 on page 74 of the notes.)

(b) Prove that $U(1)$ and $SO(2)$ are isomorphic as groups.
[Hint: use that $\mathbb{C}\setminus\{0\}$ is isomorphic to a certain subgroup of $\mathrm{GL}_2(\mathbb{R})$.]

Solution:

a) Let $A \in O(n)$ be an orthogonal matrix with column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$. Then A represents an orientation preserving map, iff the orthonormal basis $(A\mathbf{e}_1, \dots, A\mathbf{e}_n)$ is positively oriented, iff $\det(\mathbf{a}_1, \dots, \mathbf{a}_n) = 1$, iff $\det(A) = 1$.

b) Elements of $U(1)$ are essentially complex numbers $\lambda = a + bi$ with absolute value 1, with the group structure being given by multiplication. Now consider the following map:

$$\varphi : U(1) \to SO(2) : a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

This is well-defined, because $\det\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2 + b^2 = |a+bi|^2 = 1$ and furthermore it defines an injective group homomorphism.

So what remains to be shown is that φ is surjective. If $A \in O(2)$ has $\mathbf{a}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ as first column, then $|a + bi| = \|\mathbf{a}_1\| = 1$, because A is orthogonal. We claim that $A = \varphi(a + bi)$, i.e. that the second column vector $\mathbf{a}_2$ of A is uniquely determined by $\mathbf{a}_1$ (and therefore has to be $\begin{pmatrix} -b \\ a \end{pmatrix}$). Now, $\mathbf{a}_2$ has to be another unit vector in $\mathbb{R}^2$ that is orthogonal to $\begin{pmatrix} a \\ b \end{pmatrix}$ (that leaves only two possibilities), and has to be such that $(\mathbf{a}_1, \mathbf{a}_2)$ is positively oriented (which determines it completely).

Exercise T3 (Orientation preserving orthogonal maps)

(a) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be an orientation preserving orthogonal map. Show that, for any set of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n$, we have

$$\det(\varphi(\mathbf{a}_1), \ldots, \varphi(\mathbf{a}_n)) = \det(\mathbf{a}_1, \ldots, \mathbf{a}_n).$$

(b) Show that any orientation preserving orthogonal map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ preserves the cross product.

Hint: recall from Exercise (T15.2) from Linear Algebra I that the cross product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is the unique vector $\mathbf{a} \times \mathbf{b} \in \mathbb{R}^3$ such that $\langle \mathbf{a} \times \mathbf{b}, \mathbf{x} \rangle = \det(\mathbf{a}, \mathbf{b}, \mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

(c) Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be the orientation preserving orthogonal map with $\varphi(2,1,2) = (0,3,0)$ and $\varphi(0,-3,0) = (2,-1,-2)$. Determine the matrix representation of φ with respect to the standard basis, and interpret φ geometrically.

Hint: use (b).

Solution:

a) Since φ is an orientation preserving orthogonal map, $\det(\varphi) = 1$. Now let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be the unique map such that $\psi(\mathbf{e}_i) = \mathbf{a}_i$. We get that

$$\begin{aligned}
\det(\varphi(\mathbf{a}_1), \ldots, \varphi(\mathbf{a}_n)) &= \det((\varphi \circ \psi)(\mathbf{e}_1), \ldots, (\varphi \circ \psi)(\mathbf{e}_n)) \\
&= \det(\varphi \circ \psi) \\
&= \det(\varphi)\det(\psi) \\
&= \det(\psi) \\
&= \det(\mathbf{a}_1, \ldots, \mathbf{a}_n).
\end{aligned}$$

b) Since φ is an orientation preserving orthogonal map, we see that

$$\begin{aligned}
\langle \varphi(\mathbf{a} \times \mathbf{b}), \varphi(\mathbf{x}) \rangle = \langle \mathbf{a} \times \mathbf{b}, \mathbf{x} \rangle &= \det(\mathbf{a}, \mathbf{b}, \mathbf{x}) \\
&= \det(\varphi(\mathbf{a}), \varphi(\mathbf{b}), \varphi(\mathbf{x})) = \langle \varphi(\mathbf{a}) \times \varphi(\mathbf{b}), \varphi(\mathbf{x}) \rangle,
\end{aligned}$$

for any $\mathbf{x} \in \mathbb{R}^3$. By surjectivity of φ and the fact that scalar products are non-degenerate, it follows that $\varphi(\mathbf{a} \times \mathbf{b}) = \varphi(\mathbf{a}) \times \varphi(\mathbf{b})$.

c) Notice that the length of the two vectors $\mathbf{b}_1 = (2,1,2)$ and $\mathbf{b}_2 = (0,-3,0)$ as well as their scalar product is preserved by φ. This is necessary for φ to be orthogonal. We also know from (b) that φ maps $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 = (6,0,-6)$ to $\varphi(\mathbf{b}_1) \times \varphi(\mathbf{b}_2) = (-6,0,-6)$.

Note that $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is a labelled basis for $\mathbb{R}^3$. If we write $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for the standard basis of $\mathbb{R}^3$ (as usual), we get

$$[\![\varphi]\!]^B_E = \begin{pmatrix} 0 & 2 & -6 \\ 3 & -1 & 0 \\ 0 & -2 & -6 \end{pmatrix}, \qquad [\![\mathrm{id}]\!]^B_E = \begin{pmatrix} 2 & 0 & 6 \\ 1 & -3 & 0 \\ 2 & 0 & -6 \end{pmatrix},$$

$$[\![\mathrm{id}]\!]^E_B = ([\![\mathrm{id}]\!]^B_E)^{-1} = \frac{1}{12}\begin{pmatrix} 3 & 0 & 3 \\ 1 & -4 & 1 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad [\![\varphi]\!]^E_E = \frac{1}{3}\begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}.$$

The characteristic polynomial of φ is $(\lambda - 1)(-\lambda^2 - \frac{2}{3}\lambda - 1)$, which means that the eigenvalues are 1 and $\frac{1}{3}(-1 \pm 2\sqrt{2}i)$. Therefore φ is a rotation about an axis in $\mathbb{R}^3$.

Exercise T4 (Orthogonal maps)

(a) Show that an orthogonal map in $\mathbb{R}^2$ is either the identity, the reflection in the origin, a reflection in a line or a rotation (the first two being special cases of the fourth). Conclude that every orthogonal map in $\mathbb{R}^2$ is the composition of at most two reflections in a line.

(b) Show that an orthogonal map in $\mathbb{R}^3$ is either the identity, a reflection in a plane, a reflection in a line, the reflection in the origin, a rotation about an axis or a rotation about an axis followed by a reflection in the plane orthogonal to the axis (the first four being special cases of the last two). Conclude that every orthogonal map in $\mathbb{R}^3$ is the composition of at most three reflections in a plane.
Extra: how about orthogonal maps in $\mathbb{R}^n$?

[Hint: Take a look at Corollary 2.3.18 in the notes.]

Solution:

a) Corollary 2.3.18 gives the following matrix representations for an orthogonal map of $\mathbb{R}^2$ w.r.t. a suitably chosen orthonormal basis:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

So it is either the identity, the reflection in the origin, a reflection in a line or a rotation. Every rotation is the composition of two reflections in a line (the rotation through an angle α is the composition of the reflection in x-axis, followed by the reflection in the line that makes an angle $\frac{1}{2}\alpha$ with the x-axis). This implies that every orthogonal map in $\mathbb{R}^2$ is the composition of at most two reflections in a line.

b) In the same manner, we obtain the following matrix representations for an orthogonal map of $\mathbb{R}^3$ w.r.t. a suitably chosen orthonormal basis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}.$$

So an orthogonal map of $\mathbb{R}^3$ is either the identity, a reflection in a plane, a reflection in a line, the reflection in the origin, a rotation about an axis or a rotation about an axis followed by a reflection in the plane orthogonal to the axis.

We see that every orthogonal map in $\mathbb{R}^3$ is the composition of at most three reflections in a plane. In general, any orthogonal map in $\mathbb{R}^n$ is the composition of at most n reflections in $(n-1)$-dimensional hyperplanes.