Linear Algebra II
Tutorial Sheet no. 8

TECHNISCHE UNIVERSITÄT DARMSTADT

Summer term 2011
May 30, 2011

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise T1 (Warm-up)

Let V be a vector space with basis $B = (b_1, \dots, \mathbf{b}_n)$

(a) Give a definition of non-degeneracy for bilinear forms on V, and show that

   i. σ is non-degenerate iff $[\![\sigma]\!]^B$ is regular,

   ii. σ is symmetric/hermitian iff $[\![\sigma]\!]^B$ is symmetric/self-adjoint.

(b) Check for consistency that the change-of-basis transformation for matrices for bilinear forms are such that regularity, symmetry, self-adjointness are preserved.

(c) Let $\approx$ be the "similarity" of real/complex matrices as representations of the same (semi-)bilinear form. Which real/complex $n \times n$ matrices exactly are $\approx$ equivalent to the n-dimensional unit matrix?

Solution:

a) σ is non-degenerate if the condition $\sigma(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in V$ implies that $\mathbf{v} = \mathbf{0}$.

   i. Let A be a regular representation of σ in some basis. Then $\sigma(\mathbf{v}, A^{-1}\mathbf{v}) = \mathbf{v}^t AA^{-1}\mathbf{v} > 0$ for all non-null vector $\mathbf{v}$. Let A be a non-regular representation of σ in some basis. So A^t is also non regular, so let $\mathbf{v} \neq \mathbf{0}$ be such that $A^t\mathbf{v} = \mathbf{0}$. So $\sigma(\mathbf{v}, \mathbf{w}) = \mathbf{v}^t A\mathbf{w} = \mathbf{0}$ for all $\mathbf{w}$.

   ii. We do it for the symmetric case, the Hermitian case is similar. By definition, the ij-th entry of $[\![\sigma]\!]^B$ is $\sigma(b_i, b_j)$. Since $\sigma(b_i, b_j) = \sigma(b_j, b_i)$, it follows that $[\![\sigma]\!]^B$ is symmetric. Conversely, if $[\![\sigma]\!]^B$ is symmetric, it follows that $\sigma(\mathbf{v}, \mathbf{w}) = (\mathbf{w}^t [\![\sigma]\!]_B^t \mathbf{v})^t = (\sigma(\mathbf{w}, \mathbf{v}))^t = \sigma(\mathbf{w}, \mathbf{v})$.

b) For C regular, A is regular iff C^+AC is regular. If A is self-adjoint, $(C^+AC)^+ = C^+A^+C^{++} = C^+AC$.

c) The matrices C^+C where C is regular. These matrices are exactly the symmetric/self-adjoint matrices that are positive definite. (Cf chapter 3.2.3 in the notes for the latter.) In other words, they are exactly the matrices that represent a scalar product!

Exercise T2 (Orthogonal complements in $\mathbb{R}^3$)

For each of the following subspaces U in $\mathbb{R}^3$, find an orthonormal basis for U, extend this to an orthonormal basis for $\mathbb{R}^3$, and then give an orthonormal basis for $U^\perp$.

(a) $U = \{(x, y, z) \mid x + 2y + 3z = 0\}$.

(b) $U = \{(x, y, z) \mid x + y + z = 0 \text{ and } x - y + z = 0\}$.

Solution:

a) U is a plane in $\mathbb{R}^2$. The two vectors $\frac{1}{\sqrt{5}}(2, -1, 0)$ and $\frac{1}{\sqrt{70}}(3, 6, -5)$ constitute an orthonormal basis of U. We extend this basis with $\frac{1}{\sqrt{14}}(1, 2, 3)$ to obtain an orthonormal basis of $\mathbb{R}^3$. Therefore $\frac{1}{\sqrt{14}}(1, 2, 3)$ forms an orthonormal basis for $U^\perp$.

b) The subspace U is a line. Its orthogonal complement will therefore be a plane. The vector $\frac{1}{\sqrt{2}}(1, 0, -1)$ forms an orthonormal basis for U since it satisfies the equations of both planes. We extend this basis with the vectors $\frac{1}{\sqrt{2}}(1, 0, 1)$ and $(0, 1, 0)$ to obtain an orthonormal basis of $\mathbb{R}^3$. Therefore $\frac{1}{\sqrt{2}}(1, 0, 1)$ and $(0, 1, 0)$ form an orthonormal basis for $U^\perp$.

Exercise T3 (An orthonormal basis)

Let $V := \mathrm{Pol}_2(\mathbb{R})$ be the $\mathbb{R}$-vector space of all polynomial functions over $\mathbb{R}$ of degree at most 2. On this vector space

$$\langle p_1, p_2 \rangle := \int_{-1}^{1} p_1(x)\, p_2(x)\, \mathrm{d}x$$

defines a scalar product, turning $(V, \langle .\,, . \rangle)$ into a euclidean space (see Section 2.2 on page 62 of the notes).

Determine an orthonormal basis of V.

Solution:

Obviously (p_0, p_1, p_2), where $p_i(x) = x^i$ is a basis of $\mathrm{Pol}_2(\mathbb{R})$. Noting that $\int_{-1}^{1} f(x)\,\mathrm{d}x = 0$ for all odd functions f, we see that $p_1 \perp p_0, p_2$.

Using the Gram-Schmidt procedure (see Theorem 2.3.4 on page 65 of the notes) we get

$$\hat{p_0} = \frac{p_0}{\|p_0\|},$$

where $\|p_0\| = \left(\int_{-1}^{1} 1\,\mathrm{d}x \right)^{\frac{1}{2}} = \sqrt{2}$, so $\hat{p_0} = \frac{\sqrt{2}}{2}$. Further,

$$\hat{p_1} = \frac{p_1 - u}{\|p_1 - u\|},$$

where $u = \langle \hat{p_0}, p_1 \rangle \hat{p_0} = 0$. This gives

$$\hat{p_1} = \frac{p_1}{\|p_1\|},$$

where $\|p_1\| = \left(\int_{-1}^{1} x^2\,\mathrm{d}x \right)^{\frac{1}{2}} = \frac{\sqrt{6}}{3}$, so $\hat{p_1} = \frac{\sqrt{6}}{2}x$.

In the next step we get

$$\hat{p_2} = \frac{p_2 - u}{\|p_2 - u\|},$$

where

$$u = \langle \hat{p_0}, p_2 \rangle \hat{p_0} + \langle \hat{p_1}, p_2 \rangle \hat{p_1} = \left(\int_{-1}^{1} \frac{\sqrt{2}}{2} x^2\,\mathrm{d}x \right) \frac{\sqrt{2}}{2} + 0 = \frac{1}{3},$$

and

$$\|p_2 - u\| = \|x^2 - \frac{1}{3}\| = \left(\int_{-1}^{1} \left(x^2 - \frac{1}{3} \right)^2 \mathrm{d}x \right)^{\frac{1}{2}} = \frac{2\sqrt{10}}{15}.$$

Hence

$$\hat{p_2} = \frac{3\sqrt{10}}{4} \left(x^2 - \frac{1}{3} \right).$$

Now $(\hat{p_0}, \hat{p_1}, \hat{p_2})$ is an orthonormal basis.

Exercise T4 (Dual spaces)

Recall that for any $\mathbb{F}$-vector space V, the set $\mathrm{Hom}(V, \mathbb{F})$ of linear maps $V \to \mathbb{F}$ has again the structure of a vector space, with vector addition and scalar multiplication being defined pointwise, turning it into what is called the dual space of V (see Section 3.2.2 on page 87 of the notes of Linear Algebra I).

If V is a euclidean vector space, we have a map $\varphi_V : V \to \mathrm{Hom}(V, \mathbb{R})$ with $\varphi_V(\mathbf{w}) \in \mathrm{Hom}(V, \mathbb{R})$ for any $\mathbf{w} \in V$ defined by

$$\varphi_V(\mathbf{w})(\mathbf{v}) = \langle \mathbf{w}, \mathbf{v} \rangle, \text{ for all } \mathbf{v} \in V.$$

The aim of this exercise is to show that φ_V is an isomorphism if V is finite dimensional, but not necessarily if V is infinite dimensional.

(a) Show that φ_V is an injective linear map.

(b) Show that φ_V is an isomorphism if V is finite dimensional.

From now on, we consider the sequence space $\mathscr{F}(\mathbb{N},\mathbb{R})$ and define

$$V = \{f \in \mathscr{F}(\mathbb{N},\mathbb{R}) : f(n) = 0 \text{ for all but finitely many } n\}.$$

(c) Show that $\langle f, g\rangle = \sum_{n\in\mathbb{N}} f(n)g(n)$ defines a scalar product on the subspace V of $\mathscr{F}(\mathbb{N},\mathbb{R})$, turning $(V, \langle .,.\rangle)$ into a euclidean space. Check that $\langle f, g\rangle$ is defined if $f \in \mathscr{F}(\mathbb{N},\mathbb{R})$ and $g \in V$, but not necessarily if f and g both belong to $\mathscr{F}(\mathbb{N},\mathbb{R})$.

(d) Show that the map $\psi : \mathscr{F}(\mathbb{N},\mathbb{R}) \to \mathrm{Hom}(V,\mathbb{R})$ with $\psi(f) \in \mathrm{Hom}(V,\mathbb{R})$ for any $f \in \mathscr{F}(\mathbb{N},\mathbb{R})$ defined by $\psi(f)(g) = \langle f, g\rangle$ is an isomorphism of vector spaces. Conclude from this that φ_V, which is ψ restricted to V, is not.

Hint: use that the functions $\mathbf{b}_n \in V$ $(n \in \mathbb{N})$ defined by $\mathbf{b}_n(i) = 1$ if $n = i$ and 0 otherwise, form an orthonormal basis for V.

Solution:

a) That φ_V is linear follows from linearity of the scalar product in the first component in the euclidean case. If $\varphi_V(\mathbf{w}) = \varphi_V(\mathbf{w}')$, then $\varphi_V(\mathbf{w})(\mathbf{w}-\mathbf{w}') = \varphi_V(\mathbf{w}')(\mathbf{w}-\mathbf{w}')$, i.e. $\langle \mathbf{w}, \mathbf{w}-\mathbf{w}'\rangle = \langle \mathbf{w}', \mathbf{w}-\mathbf{w}'\rangle$. From this it follows that $\langle \mathbf{w}-\mathbf{w}', \mathbf{w}-\mathbf{w}'\rangle = 0$ and hence $\mathbf{w} = \mathbf{w}'$ by positive definiteness of the scalar product.

b) We only need to show that φ_V is surjective in case V is finite dimensional. Let $\gamma \in \mathrm{Hom}(V,\mathbb{F})$ and $(\mathbf{v}_1,\ldots,\mathbf{v}_n)$ be an orthonormal basis for V. We claim $\gamma = \varphi_V(\mathbf{w})$ for $\mathbf{w} = \gamma(\mathbf{v}_1)\mathbf{v}_1 + \gamma(\mathbf{v}_2)\mathbf{v}_2 + \ldots + \gamma(\mathbf{v}_n)\mathbf{v}_n$. Actually, this is an immediate consequence of orthonormality of the basis $(\mathbf{v}_1,\ldots,\mathbf{v}_n)$:

$$\begin{aligned}\langle \mathbf{w}, \mathbf{v}_i\rangle &= \langle \gamma(\mathbf{v}_1)\mathbf{v}_1 + \gamma(\mathbf{v}_2)\mathbf{v}_2 + \ldots + \gamma(\mathbf{v}_n)\mathbf{v}_n, \mathbf{v}_i\rangle \\ &= \gamma(\mathbf{v}_1)\langle \mathbf{v}_1, \mathbf{v}_i\rangle + \gamma(\mathbf{v}_2)\langle \mathbf{v}_2, \mathbf{v}_i\rangle + \ldots + \gamma(\mathbf{v}_n)\langle \mathbf{v}_n, \mathbf{v}_i\rangle \\ &= \gamma(\mathbf{v}_i),\end{aligned}$$

so $\varphi_V(\mathbf{w})(\mathbf{v}_i) = \gamma(\mathbf{v}_i)$ for every basis vector $\mathbf{v}_i$, and therefore $\gamma_V(\mathbf{w}) = \gamma$.

c) For simplicity, let us denote $W := \mathscr{F}(\mathbb{N},\mathbb{R})$. For any $f \in W$, let

$$S(f) = \{n \in \mathbb{N} : f(n) \neq 0\}$$

be the *support* of f. Then

$$V = \{f \in W : S(f) \text{ is finite}\}.$$

Since $S(0) = \emptyset$, $S(\lambda f) \subseteq S(f)$ and $S(f+g) \subseteq S(f) \cup S(g)$, it is easy to see that V is a subspace of W.

In general, $\langle f, g\rangle = \sum_{n\in\mathbb{N}} f(n)g(n)$ does not converge for $f, g \in W$ (for example, if f and g are both constant 1), but in case $g \in V$, we get that

$$\sum_{n\in\mathbb{N}} f(n)g(n) = \sum_{n\in S(g)} f(n)g(n) < \infty, \quad \text{as } S(g) \text{ is finite.}$$

That $\langle .,.\rangle$ defines a scalar product on V is then clear: it is linear in both components, symmetric and positive definite.

d) Linearity of ψ is clear. It remains to show that it is bijective. Let us define for every $n \in \mathbb{N}$,

$$\mathbf{b}_n : \mathbb{N} \to \mathbb{R}, \quad \mathbf{b}_n(i) = \begin{cases} 1 & \text{if } n = i \\ 0 & \text{otherwise.}\end{cases}$$

Then $\mathbf{b}_n$ is a unit vector in V for all n and it is easy to see that the $\mathbf{b}_n$ are pairwise orthogonal, hence linearly independent. Moreover, they span V, since

$$f = \sum_{n\in S(f)} f(n)\mathbf{b}_n \quad \text{for any } f \in V.$$

Thus, $B = (\mathbf{b}_n)_{n\in\mathbb{N}}$ is an orthonormal basis of V.

Let us define now $\chi : \mathrm{Hom}(V,\mathbb{R}) \to W$ by

$$\chi(\gamma)(n) = \gamma(\mathbf{b}_n) \quad \text{for all } \gamma \in \mathrm{Hom}(V,\mathbb{R}), n \in \mathbb{N}.$$

Then for any $g \in W$, we get that

$$(\chi \circ \psi)(g)(n) \quad = \quad \chi(\psi(g))(n) = \psi(g)(\mathbf{b}_n) = \langle g, \mathbf{b}_n \rangle = g(n), \quad \text{so } \chi \circ \psi = \mathrm{id}_W.$$

Furthermore, for any $\gamma \in \mathrm{Hom}(V,\mathbb{R})$,

$$(\psi \circ \chi)(\gamma)(\mathbf{b}_n) = \psi(\chi(\gamma))(\mathbf{b}_n) = \langle \chi(\gamma), \mathbf{b}_n \rangle = \chi(\gamma)(n) = \gamma(\mathbf{b}_n) \text{ for all } n \in \mathbb{N}.$$

Since $B = (\mathbf{b}_n)_{n\in\mathbb{N}}$ is a basis of V, we conclude that $(\psi \circ \chi)(\gamma) = \gamma$ for all $\gamma \in \mathrm{Hom}(V,\mathbb{R})$, therefore $\psi \circ \chi = \mathrm{id}_{\mathrm{Hom}(V,\mathbb{R})}$.

Thus, $\psi : W \to \mathrm{Hom}(V,\mathbb{R})$ is an isomorphism of vector spaces. Since φ_V is the restriction of ψ to V and V is a proper subspace of W, it follows that φ_V cannot be surjective.