Linear Algebra II
Tutorial Sheet no. 7

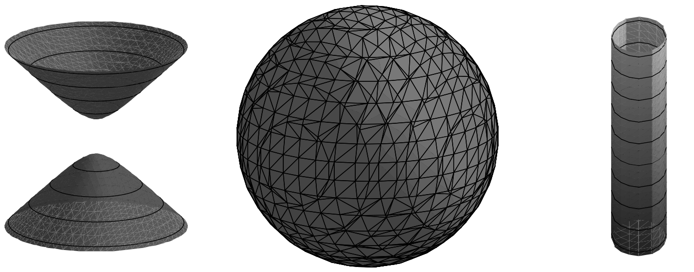
TECHNISCHE UNIVERSITÄT DARMSTADT

Summer term 2011
May 24, 2011

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise T1 (Warm up: bilinear form - examples)

Each of the following pictures shows the unit surface $\{v \in \mathbb{R}^3 : \sigma(\mathbf{v}, \mathbf{v}) = 1\}$ for some bilinear form σ.
Which of these forms is positive definite?
Which are non-degenerate?
For each of these bilinear forms, give an example of a matrix that represents that form.



Solution:

Picture 1 The bilinear form is non-degenerate and is not positive definite. An example of a matrix representing this form is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Picture 2 The bilinear form is non-degenerate and positive definite. An example of a matrix representing this form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Picture 3 The bilinear form is degenerate and not positive definite. An example of a matrix representing this form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Exercise T2 (Basis transformations for (semi-)bilinear forms)

Compare Exercise 2.1.3 on page 58 of the notes.

(a) Let σ be a bilinear form on an n-dimensional $\mathbb{R}$-vector space V, represented by the matrix A with respect to the basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$. If $B' = (\mathbf{b}'_1, \dots, \mathbf{b}'_n)$ is another basis for V, find an expression for the matrix $A' = [\![\sigma]\!]^{B'}$ in terms of A, the basis transformation matrices $C = [\![\mathrm{id}_V]\!]^{B}_{B'}$, and $C^{-1} = [\![\mathrm{id}_V]\!]^{B'}_{B}$ as well as their transposes as appropriate.

(b) Similarly for a semi-bilinear form σ of an n-dimensional $\mathbb{C}$-vector space V: if σ is represented by A w.r.t. a basis B, what is its representation A' w.r.t. a basis B' in terms of A, the basis transformations matrices, as well as their adjoints?

(c) Consider the following bilinear form $\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ on the vector space $\mathbb{R}^2$:

$$\sigma(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}) := 7v_1w_1 - 5v_1w_2 - 5v_2w_1 + 4v_2w_2.$$

What is its representation with respect to the standard basis? Then compute its representation with respect to the basis $(\mathbf{b}_1, \mathbf{b}_2) = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix})$ directly, as well as by using the formula obtained in part (a).

(d) Is the bilinear form σ in part (c) symmetric? Is it positive definite?

Solution:

a) $\sigma(\mathbf{u}, \mathbf{v}) = [\![\mathbf{u}]\!]_B^t A [\![\mathbf{v}]\!]_B = (C^{-1}[\![\mathbf{u}]\!]_{B'})^t A C^{-1}[\![\mathbf{v}]\!]_{B'} = [\![\mathbf{u}]\!]_{B'}^t (C^{-1})^t A C^{-1}[\![\mathbf{v}]\!]_{B'}$, so $A' = (C^{-1})^t A C^{-1}$.

b) $\sigma(\mathbf{u}, \mathbf{v}) = [\![\mathbf{u}]\!]_B^+ A [\![\mathbf{v}]\!]_B = (C^{-1}[\![\mathbf{u}]\!]_{B'})^+ A C^{-1}[\![\mathbf{v}]\!]_{B'} = [\![\mathbf{u}]\!]_{B'}^+ (C^{-1})^+ A C^{-1}[\![\mathbf{v}]\!]_{B'}$, so $A' = (C^{-1})^+ A C^{-1}$.

c) Its representation with respect to the standard basis is $\begin{pmatrix} 7 & -5 \\ -5 & 4 \end{pmatrix}$, and with respect to $(\mathbf{b}_1, \mathbf{b}_2)$ is $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

d) It is both: symmetry can be proved directly, or by observing that both matrix representations are symmetric. That it is positive definite follows from the second representation: if $\mathbf{v} = \lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2$, then $\sigma(\mathbf{v}, \mathbf{v}) = \lambda_1^2 + 3\lambda_2^2$. So $\sigma(\mathbf{v}, \mathbf{v}) \geqslant 0$ and $\sigma(\mathbf{v}, \mathbf{v}) = 0$ iff $\lambda_1 = \lambda_2 = 0$ iff $\mathbf{v} = 0$.

Exercise T3

Let $\mathscr{F}(\mathbb{N}, \mathbb{C})$ denote the set of sequences $\mathbf{a} = (a_0, a_1, a_2, \dots)$ in $\mathbb{C}$.

(a) Show that given $\mathbf{a}, \mathbf{b}$ in $\mathscr{F}(\mathbb{N}, \mathbb{C})$ for which $\sum_{i=0}^{\infty} |a_i|^2$ and $\sum_{i=0}^{\infty} |b_i|^2$ are convergent, $\sum_{i=0}^{\infty} \bar{a}_i b_i$ is absolutely convergent.

(b) Let $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$ denote the set of sequences $\mathbf{a} \in \mathscr{F}(\mathbb{N}, \mathbb{C})$ for which $\sum_{i=0}^{\infty} |a_i|^2$ converges. Show that $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$ forms a subspace of $\mathscr{F}(\mathbb{N}, \mathbb{C})$, and that the scalar product $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=0}^{\infty} \bar{a}_i b_i$ on $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$ is unitary.

(c) Show that the "generalised standard basis vectors" consisting of sequences with a single 1 and zeroes elsewhere form an infinite family of pairwise orthogonal unit vectors in $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$, but they do not form a basis of $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$.

(d) (Cf Exercise H10.3 on the Christmas sheet LA I) Does $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$ admit a countable basis?

Solution:

a) Fix $m > 0$ and let $\mathbf{v} \in \mathbb{C}^{m+1}$ be the vector $(|a_0|, \dots, |a_m|)$. Similarly, let $\mathbf{w} = (|b_0|, \dots, |b_m|)$. By the Cauchy-Schwarz inequality in $\mathbb{C}^{m+1}$ (Proposition 2.1.10, p. 59 of the lecture notes) we have

$$(\sum_{i=0}^{m} |a_i||b_i|)^2 \leqslant \sum_{i=0}^{m} |a_i|^2 \sum_{i=0}^{m} |b_i|^2.$$

Since $|\bar{a}_i b_i| = |a_i||b_i|$, this implies that

$$\sum_{i=0}^{m} |\bar{a}_i b_i| \leqslant \left(\sum_{i=0}^{m} |a_i|^2 \sum_{i=0}^{m} |b_i|^2 \right)^{1/2}.$$

Taking the limit as m approaches ∞ yields the absolute convergence of $\sum_{i=0}^{\infty} \bar{a}_i b_i$.

b) Let $\mathbf{a}, \mathbf{b} \in \mathscr{F}(\mathbb{N}, \mathbb{C})_2$. Clearly $\lambda\mathbf{a}$ lies in $\mathscr{F}(\mathbb{N}, \mathbb{C})_2$ for any $\lambda \in \mathbb{C}$. To see that $\mathbf{a} + \mathbf{b} \in \mathscr{F}(\mathbb{N}, \mathbb{C})_2$ as well, note that

$$\sum_{i=0}^{\infty} |a_i + b_i|^2 = \sum_{i=0}^{\infty} |a_i|^2 + \sum_{i=0}^{\infty} |b_i|^2 + \sum_{i=0}^{\infty} \bar{a}_i b_i + \sum_{i=0}^{\infty} \bar{b}_i a_i.$$

By Part (a) above, both $\sum_{i=0}^{\infty} \bar{a}_i b_i$ and $\sum_{i=0}^{\infty} \bar{b}_i a_i$ are absolutely convergent, so this sum converges. It follows that $\mathbf{a} + \mathbf{b} \in \mathscr{F}(\mathbb{N}, \mathbb{C})_2$. Unitarity is clear from the fact that

$$\langle \mathbf{b}, \mathbf{a} \rangle = \sum_{i=0}^{\infty} \bar{b}_i a_i = \overline{\sum_{i=0}^{\infty} \bar{a}_i b_i} = \overline{\langle \mathbf{a}, \mathbf{b} \rangle}.$$

c) Let $\mathbf{e}_i$ denote the sequence for which $a_i = 1$ and $a_j = 0$ for all $j \neq i$. The vector spaces spanned by $\{\mathbf{e}_i : i \geqslant 0\}$ consists of all sequences which possess only finitely many non-zero terms. (These are precisely the $\mathbb{C}$-linear combinations of the basis vectors). Clearly for $i \neq j$, we have $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$, so these vectors are pairwise orthogonal. To see that they do not form basis, note that $\mathscr{F}(\mathbb{N},\mathbb{C})_2$ contains elements $\mathbf{a}$ which contain infinitely many nonzero elements. An example is the sequence $\mathbf{a}$ defined by $a_i = \frac{1}{2^i}$.

d) We claim that $\mathscr{F}(\mathbb{N},\mathbb{C})_2$ does *not* admit a countable basis. Let $(\mathbf{v}^0, \mathbf{v}^1, \dots)$ be any subset of $\mathscr{F}(\mathbb{N},\mathbb{C})_2$ indexed by the natural numbers. Using a procedure similar to Cantor's diagonalization, we will construct an element $\mathbf{u} \in \mathscr{F}(\mathbb{N},\mathbb{C})_2$ which cannot be expressed as a linear combination of the elements $(\mathbf{v}^0, \mathbf{v}^1, \dots)$. This shows that this collection does not span $\mathscr{F}(\mathbb{N},\mathbb{C})_2$, and therefore is not a basis.

First, we define a partition of $\mathbb{N}$ consisting of sets

$$X_n := \{n^2, n^2+1, \dots, (n+1)^2 - 1\}, \qquad n \geqslant 0.$$

Clearly X_n contains $2n+1$ elements, and $\cup_{n\in\mathbb{N}} X_n = \mathbb{N}$ and $X_i \cap X_j = \emptyset$ for $i \neq j$. Given $\mathbf{u} = (u_0, u_1, \dots)$ in $\mathscr{F}(\mathbb{N},\mathbb{C})_2$, we denote by $R_n(\mathbf{u})$ the restriction of $\mathbf{u}$ to the subset $X_n \subseteq \mathbb{N}$. For restriction to make sense, we are regarding our sequence $\mathbf{u}$ as a function from $\mathbb{N} \to \mathbb{C}$, and $R_n(\mathbf{u})$ consists of the finite ordered set $(u_{n^2}, u_{n^2+1}, \dots, u_{(n+1)^2-1})$. Since $\{X_n : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$, a sequence $\mathbf{u} \in \mathscr{F}(\mathbb{N},\mathbb{C})_2$ is specified by its restrictions $\{R_n(\mathbf{u}) : n \in \mathbb{N}\}$. Conversely, a family of finite sequences $\tilde{u}_n = (u_{n^2}, u_{n^2+1}, \dots, u_{(n+1)^2-1})$ specifies an infinite sequence $\mathbf{u} = (u_0, u_1, \dots)$, such that $R_n(\mathbf{u}) = \tilde{u}_n$. If furthermore, we choose u_i so that $|u_i| < \frac{1}{2^i}$ for all i, the sequence $\mathbf{u}$ will lie in $\mathscr{F}(\mathbb{N},\mathbb{C})_2$.

For each $n \in \mathbb{N}$, the collection $(R_n(\mathbf{v}^1), \dots, R_n(\mathbf{v}^n))$ spans a vector space of dimension at most n. Since the set of all sequences of the form $\tilde{u}_n = (u_{n^2}, u_{n^2+1}, \dots, u_{(n+1)^2-1})$ has dimension $2n+1$, we can choose some $\tilde{u}_n$ which does *not* lie in the span of $(R_n(\mathbf{v}^1), \dots, R_n(\mathbf{v}^n))$. Moreover, by rescaling $\tilde{u}_n$ if necessary, we may assume that each $|u_i| < \frac{1}{2^i}$. Let $\mathbf{u} = (u_0, u_1 \dots)$ be the sequence such that $R_n(\mathbf{u}) = \tilde{u}_n$, which clearly lies in $\mathscr{F}(\mathbb{N},\mathbb{C})_2$. Finally, suppose that $\mathbf{u}$ lies in the span of $(\mathbf{v}^0, \mathbf{v}^1, \dots)$, so that $\mathbf{u} = \sum_{i=0}^{k} \lambda_i \mathbf{v}^{\mathbf{i}}$ for some k. But this is impossible because $\tilde{u}_k = R_k(\mathbf{u})$ does not lie in the span of $(R_k(\mathbf{v}^1), \dots, R_k(\mathbf{v}^k))$, by construction.

Exercise T4

In $\mathbb{R}^3$, let ρ be the rotation through an angle of $\frac{\pi}{4}$ about the vector $(1,1,1)$. In this problem we will find the matrix representing ρ w.r.t. the standard basis of $\mathbb{R}^3$.

(a) Find an orthonormal basis $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ of $\mathbb{R}^3$ such that $\mathbf{b}_1$ is a positive scalar multiple of $(1,1,1)$ and $[\![\mathrm{id}]\!]^B_S$ is an orthogonal matrix of determinant 1.

(b) Write down the matrix representing ρ w.r.t. the basis B.

(c) Express the matrix representing ρ using $[\![\rho]\!]^B_B$ and $[\![\mathrm{id}]\!]^B_S$.

Solution:

a) We can take $\mathbf{b}_1 = \frac{1}{\sqrt{3}}(1,1,1)$, $\mathbf{b}_2 = \frac{1}{\sqrt{6}}(-1,2,-1)$, $\mathbf{b}_3 = \frac{1}{\sqrt{2}}(-1,0,1)$ and

$$[\![\mathrm{id}]\!]^B_S = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{6} & -1\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1\sqrt{2} \end{pmatrix}.$$

b)

$$[\![\rho]\!]^B_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.$$

c) $[\![\rho]\!]^S_S = [\![\mathrm{id}]\!]^B_S [\![\rho]\!]^B_B [\![\mathrm{id}]\!]^S_B$.