Linear Algebra II
Tutorial Sheet no. 6

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Exercise T1 (Warm-up)

Let φ be an endomorphism of an n-dimensional $\mathbb{F}$-vector space V. Assume that φ is represented in the basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ by the matrix in block upper triangle form $M = \begin{pmatrix} A & D \\ 0 & C \end{pmatrix}$ where $A \in \mathbb{F}^{(k,k)}$, $C \in \mathbb{F}^{(n-k,n-k)}$, and $D \in \mathbb{F}^{(k,n-k)}$.

Discuss what this implies about

(a) the existence of invariant subspaces for φ.

(b) the characteristic polynomial of φ.

(c) the minimal polynomial of φ.

Consider examples of various situations of this kind.

Solution:

a) The subspace of V spanned by $(\mathbf{b}_1, \dots, \mathbf{b}_k)$ is clearly invariant under φ.

b) As already done in T13.3 LA I, let us prove that $|M| = |A||C|$, from which it is clear that, in the same way, $p_M = |M - \lambda E_n| = |A - \lambda E_k||C - \lambda E_{n-k}| = p_A p_C$. We use the following explicit formula for the determinant of $M = (m_{ij})$:

$$|M| = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) m_{\sigma(1)1} m_{\sigma(2)2} \cdots m_{\sigma(n)n}$$

(see Proposition 4.1.9 on page 121 of the lectures notes for Linear Algebra I). Because of the specific form of the matrix, the only non-zero contributions can come from $\sigma \in S_n$ that permutes $\{1, \dots, k\}$. Because σ is a permutation (and therefore a bijection) of $\{1, \dots, n\}$ it follows that σ also permutes $\{k+1, \dots, n\}$. So:

$$\begin{aligned} |M| &= \sum_{\sigma \in S_k, \sigma' \in S_{n-k}} \operatorname{sign}(\sigma)\operatorname{sign}(\sigma') m_{\sigma(1)1} \cdots m_{\sigma(k)k} m_{k+\sigma'(1),k+1} \cdots m_{k+\sigma'(n-k),n} \\ &= \left(\sum_{\sigma \in S_k} \operatorname{sign}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(k)k} \right) \left(\sum_{\sigma' \in S_{n-k}} \operatorname{sign}(\sigma') c_{\sigma'(1)1} \cdots c_{\sigma'(n-k),n-k} \right) \\ &= |A||C|. \end{aligned}$$

c) The same equality does not hold for the minimal polynomials, because for a matrix of the form $M = \begin{pmatrix} A & D \\ 0 & C \end{pmatrix}$, the content of the matrix D is important for determining the minimal polynomial of M; compare the minimal polynomials of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. What one can say is that q_M is a common multiple of q_A and q_C, but nothing more (in case $D = 0$ it is the least common multiple, see E6.2 in LA II this week). In this respect the minimal polynomial is a finer measure than the characteristic polynomial.

Exercise T2 (Jordan normal form)

Write down matrices $A_i \in \mathbb{R}^{(4,4)}$ in Jordan normal form with the following properties:

(a) A_1 has eigenvalues 2 and 4, with 2 having algebraic multiplicity 3 and geometric multiplicity 1.

(b) A_2 has the eigenvalue 5 with algebraic multiplicity 4 and geometric multiplicity 3.

(c) A_3 has the eigenvalue 7 with algebraic multiplicity 2 and geometric multiplicity 2 and the eigenvalue -3 with algebraic multiplicity 2 and geometric multiplicity 1.

(d) The matrices A_4 and A_5 both have the eigenvalue 1 with algebraic multiplicity 4 and geometric multiplicity 2 and have no other eigenvalues. Furthermore, A_4 and A_5 are not similar.

(e) Find two matrices that have the same characteristic and minimal polynomial, yet are not similar.

Solution:
Possible solutions are:

a) $A_1 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$

b) $A_2 = \begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$

c) $A_3 = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$

d) $A_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $A_5 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

e) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ don't have the same rank so they are not similar. Yet they have the same characteristic polynomial X^4 and the same minimal polynomial X^2.

Exercise T3 (Jordan normal form and transpose)

(a) Show that if the $A, B \in \mathbb{F}^{(n,n)}$ are similar, then so are A^t and B^t.

(b) Let $A \in \mathbb{F}^{(n,n)}$ be a matrix in Jordan normal form. Show that A is similar to A^t. Deduce that over $\mathbb{C}$ every square matrix is similar to its transpose.

Solution:

a) Note first that if S is regular, then so is S^t, with $(S^t)^{-1} = (S^{-1})^t$: if $SS^{-1} = E_n$, then $(S^{-1})^t S^t = E_n$. So if $A = SBS^{-1}$ for some regular S, then $A^t = (S^{-1})^t B^t S^t = (S^t)^{-1} B^t S^t$.

b) Suppose an endomorphism of an n-dimensional vector space $\varphi : V \to V$ is represented with respect to a basis $(\mathbf{b}_1, \dots, \mathbf{b}_n)$ by a matrix A consisting of a single Jordan block, as in

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \qquad A^t = \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix}.$$

Then the endomorphism $\psi : V \to V$ represented by A^t with respect to the basis $(\mathbf{b}_1, \dots, \mathbf{b}_n)$, is represented by A with respect to the basis $(\mathbf{b}_n, \dots, \mathbf{b}_1)$ (the same vectors in reverse order). This idea generalises to matrices in Jordan normal form (by reversing the order of the basis vectors for every Jordan block), proving that every matrix in Jordan normal form is similar to its transpose.

As over $\mathbb{C}$ every matrix is similar to one in Jordan normal form, this shows that every square matrix over $\mathbb{C}$ is similar to its transpose.

Exercise T4 (Square roots)

Consider the set of all 4×4 complex matrices A with characteristic polynomial $p_A(x) = X^4$. We wish to determine exactly which such matrices admit a square root, that is, some matrix S such that $S^2 = A$.

(a) Suppose that A and B are similar matrices. Show that A has a square root if and only if B has a square root. Conclude that it is enough to consider matrices that are in Jordan normal form.

(b) Show that neither of the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

admits a square root.

(c) Show that the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

admit square roots. Conclude that the set of 4×4 matrices with characteristic polynomial X^4 which admit a square root are precisely the ones which are similar to one of these Jordan forms.

Solution:

a) Suppose that $S^2 = A$ and that $B = C^{-1}AC$ for some C. Then $(C^{-1}SC)^2 = C^{-1}S^2C = C^{-1}AC = B$, so B has a square root. The situation is clearly symmetric in A and B, so the converse follows immediately. Since every matrix is similar to a matrix in Jordan form, it suffices to determine which matrices in Jordan form admit square roots.

b) The first matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is nilpotent of index 4, that is, $A^4 = 0$ but $A^3 \neq 0$. Suppose that $S^2 = A$. Then $S^8 = A^4 = 0$, so S is nilpotent, but $S^6 = A^3 \neq 0$, so S must be nilpotent of index at least 7. This is impossible since a nilpotent 4×4 matrix has index at most 4. Similarly, $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ satisfies $B^2 \neq 0$ but $B^3 = 0$. If $S^2 = B$, we must have S nilpotent and $S^4 = B^2 \neq 0$, which is impossible.

c) The zero matrix obviously admits a square root, namely itself. Note that $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, which is easily seen to be similar to $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Therefore it follows from part (a) that $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ admits a square root.

Similarly, $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, which is Jordan equivalent to $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.