Linear Algebra II
Tutorial Sheet no. 5

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Exercise T1 (Polynomials of matrices and linear maps)

In the following p,q stand for polynomials in $\mathbb{F}[X]$, φ for an endomorphism of an n-dimensional $\mathbb{F}$-vector space V, A,B for $n \times n$ matrices over $\mathbb{F}$. Which of these claims are generally true, which are false in general (and which are plain nonsense)?

(a) $p(AB) = p(A)p(B)$ (?)

(b) $(pq)(A) = p(A)q(A) = q(A)p(A)$ (?)

(c) $(p(\varphi))(\mathbf{v}) = p(\varphi(\mathbf{v}))$ (?)

(d) $[\![p(\varphi)]\!]^B_B = p([\![\varphi]\!]^B_B)$ (?)

(e) A regular $\Rightarrow p(A)$ regular (?)

(f) $A \sim B \Rightarrow p(A) \sim p(B)$ (?)

(g) $\varphi(\mathbf{v}) = \lambda\mathbf{v} \Rightarrow (p(\varphi))(\mathbf{v}) = p(\lambda)\mathbf{v}$ (?)

(h) $p(A)q(A) = 0 \Rightarrow (p(A) = 0 \vee q(A) = 0)$ (?)

(i) φ and $p(\varphi)$ have the same invariant subspaces (?)

(j) $U \subseteq V$ an invariant subspace of $\varphi \quad \Rightarrow \quad U$ invariant under $p(\varphi)$ (?)

(k) $U \subseteq V$ an invariant subspace of $\varphi \quad \Rightarrow \quad (p(\varphi))(\mathbf{v} + U) = (p(\varphi))(\mathbf{v}) + U$ (?) (φ viewed as a map on subsets of V.)

(l) $U \subseteq V$ an invariant subspace of $\varphi' \quad \Rightarrow \quad (p(\varphi'))(\mathbf{v} + U) = (p(\varphi))(\mathbf{v}) + U$ (?) (φ' the induced endomorphism of V/U.)

Solution:

a) False in general, even if A and B commute.

b) True, $p \mapsto p(A)$ is a ring homomorphism.

c) Nonsense!

d) True.

e) False, for instance $p_A(A) = 0$ is not regular.

f) True.

g) True.

h) False, consider $p = q = X$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

i) False, consider p_φ.

j) True.

k) False, consider $\varphi = 0$.

l) True.

Exercise T2 (Eigenvectors)

Consider the matrices $A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{pmatrix}$ and $B := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}$

(a) Determine the characteristic and minimal polynomials of A and B.

(b) For the matrix B:

i. Show that $\mathbf{v}_1 = (1,0,0,0)$ and $\mathbf{v}_2 = (0,0,1,1)$ are eigenvectors with eigenvalue 2.

ii. Determine an eigenvector $\mathbf{v}_4$ with eigenvalue 3.

iii. Check that $\mathbf{v}_3 = (0,1,0,0)$ is a solution of $(B-2E_4)^2\mathbf{x} = \mathbf{0}$ and that $B\mathbf{v}_3 = 2\mathbf{v}_3 + \mathbf{v}_1$.

iv. Determine the matrix that represents φ_B w.r.t. the basis $(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_4)$.

Solution:

a) For A we obtain $p_A = \begin{vmatrix} 1-X & 2 & 3 \\ 2 & 1-X & 3 \\ 3 & 3 & 6-X \end{vmatrix} = -X(X+1)(X-9)$.

Since this polynomial splits into distinct linear factors, the minimal polynomial of A is equal to $-p_A$ by Proposition 1.5.2 on page 33 of the notes. (Recall that the minimal polynomial is normalised.)

For B we obtain $p_B = \begin{vmatrix} 2-X & 1 & 0 & 0 \\ 0 & 2-X & 0 & 0 \\ 0 & 0 & 1-X & 1 \\ 0 & 0 & -2 & 4-X \end{vmatrix} = \begin{vmatrix} 2-X & 1 \\ 0 & 2-X \end{vmatrix} \begin{vmatrix} 1-X & 1-2 & 4-X \end{vmatrix} = (2-X)^3(X-3)$.

The minimal polynomial q_B has the same linear factors as the characteristic polynomial p_B. So q_B has to be one of the following: $(X-2)(X-3)$, or $(X-2)^2(X-3)$, or p_B.

As $(B-2E_4)(B-3E_4) \neq 0$ and $(B-2E_4)^2(B-3E_4) = 0$, we conclude that $q_B = (X-2)^2(X-3)$.

b) For the matrix B:

i. Since $B - 2E_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$ we have

$(B-2E_4)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$ and $(B-2E_4)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0}$.

ii. We are looking for a non-trivial solution of the equation $(B-3E_4)\mathbf{v}_4 = 0$:

$$\left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So one solution is $\mathbf{v}_4 = (0,0,1,2)$.

iii. We obtain $(B-2E_4)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$.

Hence $(B-2E_4)^2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$. Furthermore $B\mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = 2\mathbf{v}_3 + \mathbf{v}_1$.

iv. We obtain $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

Exercise T3 (Complexification)

For $A \in \mathbb{R}^{(2,2)}$ consider the associated endomorphisms $\varphi_A^{\mathbb{R}}$ and $\varphi_A^{\mathbb{C}}$, which are represented by A w.r.t. the standard bases of $\mathbb{R}^2$ and of $\mathbb{C}^2$, respectively.

Let the characteristic polynomial p_A be irreducible in $\mathbb{R}[X]$.

(a) Show that p_A has a pair of complex conjugate zeroes. (Recall that the complex conjugate of $z = \alpha + i\beta$ is $\bar{z} = \alpha - i\beta$.)

(b) Show that $\mathbb{C}^2$ has a basis $B = (\mathbf{v}, \bar{\mathbf{v}})$ of eigenvectors of φ_A consisting of a vector $\mathbf{v}$ with eigenvalue λ, and its complex conjugate $\bar{\mathbf{v}}$, which has eigenvalue $\bar{\lambda}$.

(c) Let $\mathbf{b}_1 = \frac{1}{2}(\mathbf{v} + \bar{\mathbf{v}})$ and $\mathbf{b}_2 = \frac{1}{2i}(\mathbf{v} - \bar{\mathbf{v}})$, which lie in $\mathbb{R}^2$.

  i. Show that $B' = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^2$.

  ii. Determine the matrix representation of $\varphi_A^{\mathbb{R}}$ w.r.t. basis B' and discuss the similarity of A with a matrix that would suggest the interpretation as "rotation followed by dilation"

Solution:

a) The characteristic polynomial is a quadratic of the form $x^2 + tx + d$, where $t = \mathrm{tr}(A)$ and $d = \det(A)$. By the quadratic formula, $x = \frac{-t \pm \sqrt{t^2 - 4d}}{2}$. Since $p_A(x)$ is irreducible in $\mathbb{R}[X]$ we must have $t^2 - 4d < 0$, so it is clear that the roots are distinct and occur as a complex conjugate pair.

b) Let λ and $\bar{\lambda}$ be the eigenvalues of φ_A, and let $\mathbf{v}$ be an eigenvector of φ_A with eigenvalue λ, so that $A\mathbf{v} = \lambda\mathbf{v}$. Taking complex conjugates of both sides and noting that $\bar{A} = A$ since A is real, we have $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$. Hence $\bar{\mathbf{v}}$ is eigenvector of φ_A with eigenvalue $\bar{\lambda}$. Finally, let $B = \{\mathbf{v}, \bar{\mathbf{v}}\}$. The fact that B is a basis for $\mathbb{C}^2$ is clear from the fact that the corresponding eigenvalues λ and $\bar{\lambda}$ are distinct.

c) i. First we regard $\mathbf{b}_1$ and $\mathbf{b}_2$ as elements of $\mathbb{C}^2$. It is clear that they form a basis of $\mathbb{C}^2$ because they are related to $\mathbf{v}$ and $\bar{\mathbf{v}}$ via the invertible matrix

$$[\![\mathrm{id}]\!]_B^{B'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix}.$$

Since $\mathbf{b}_1$ and $\mathbf{b}_2$ are linearly independent over $\mathbb{C}$, they must be linearly independent over $\mathbb{R}$ as well. Hence if we regard them as elements of $\mathbb{R}^2$, they form a basis of $\mathbb{R}^2$.

ii. We have $[\![\varphi_A]\!]_{B'}^{B'} = [\![\mathrm{id}]\!]_{B'}^{B} \circ [\![\varphi_A]\!]_B^B \circ [\![\mathrm{id}]\!]_B^{B'}$. Also, note that $[\![\varphi_A]\!]_B^B = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$ and $[\![\mathrm{id}]\!]_{B'}^{B} = ([\![\mathrm{id}]\!]_B^{B'})^{-1} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. We therefore obtain

$$[\![\varphi]\!]_{B'}^{B'} = \begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix} = r \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix},$$

where $\lambda = re^{i\theta}$.

Exercise T4 (Simultaneous diagonalisation and polynomials)

Let $A \in \mathbb{R}^{(n,n)}$ be a matrix with n distinct real eigenvalues, and let $B \in \mathbb{R}^{(n,n)}$ be an abitrary matrix such that A and B are simultaneously diagonalisable. Show that there exists a polynomial $p \in \mathbb{R}[X]$ such that $B = p(A)$.

Hint. Recall that, last semester in Linear Algebra I, we have shown in exercise (E14.2) that, given n distinct real numbers $a_1, \ldots, a_n \in \mathbb{R}$ and n arbitrary real numbers $b_1, \ldots, b_n \in \mathbb{R}$, there exists a polynomial p of degree $n-1$ such that $p(a_i) = b_i$ for all i.

Solution:

By assumption, there exists a matrix C such that $D := C^{-1}AC$ and $H := C^{-1}BC$ are both diagonal matrices. Suppose that

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \text{ and } H = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}.$$

Note that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Since these are all distinct, we can use the hint to find a polynomial p such that $p(\lambda_i) = \mu_i$. It follows that $p(D) = H$. Since $(CDC^{-1})^k = CD^kC^{-1}$ it follows that $p(A) = p(CDC^{-1}) = Cp(D)C^{-1} = CHC^{-1} = B$.