Linear Algebra II
Tutorial Sheet no. 4

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Exercise T1 (Algebraic and geometric multiplicity)

Let φ be an endomorphism on a finite dimensional $\mathbb{F}$-vector space V and $\lambda \in \mathbb{F}$ an eigenvalue of φ with geometric multiplicity d and algebraic multiplicity s. Show that $d \leq s$.

Hint: Choose a basis B of V that contains d eigenvectors of φ with eigenvalue λ.

Solution:

We choose a basis $\mathbf{v}_1, \cdots, \mathbf{v}_d$ for the eigenspace V_λ and extend it to a basis $\mathbf{v}_1, \cdots, \mathbf{v}_n$ for V, where n is the dimension of V. The matrix of φ with respect to this basis is a block upper tringular matrix of the form

$$A = [\![\varphi]\!]_B^B = \left(\begin{array}{c|c} \lambda E_d & B \\ \hline 0 & C \end{array}\right),$$

where E_d is the $d \times d$-unity matrix, $B \in \mathbb{F}^{(d,n-d)}$ and $C \in \mathbb{F}^{(n-d,n-d)}$. By successive expansion w.r.t. first, second, .. d-th column we find $p_\varphi = (\lambda - x)^d \det(C - xE) = (\lambda - x)^d p_C$. Indeed

$$A - xE_n = \left(\begin{array}{ccc|c} \lambda - x & & 0 & \\ & \ddots & & B \\ 0 & & \lambda - x & \\ \hline & 0 & & C - xE \end{array}\right)$$

Hence, $(\lambda - x)^d$ is a divisor of the characteristic polynomial p_φ and we conclude that $s \geq d$, where s is the algebraic multiplicity of the eigenvalue λ.

Exercise T2 (Upper triangle shape)

Find a real upper triangular matrix similar to

$$A = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Solution:

Consider $A_1 = A$ as the representation of a linear endomorphism φ_1 of $\mathbb{R}^3$ with respect to the standard basis $B_1 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The corresponding characteristic equation

$$p_{\varphi_1} = p_{A_1} = \det \begin{pmatrix} 3-\lambda & 0 & -2 \\ -2 & -\lambda & 1 \\ 2 & 1 & -\lambda \end{pmatrix} = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = (1-\lambda)^3 = 0,$$

has $\lambda_1 = 1$ as only solution. A corresponding eigenvector can be found by solving the homogeneous system of equations $(A_1 - E_3)\mathbf{v} = 0$. A possible result is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

We extend $\mathbf{v}_1$ to a new basis, for example $B_2 = (\mathbf{v}_1, \mathbf{e}_2, \mathbf{e}_3)$. The transition matrix and its inverse are given by

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

We obtain

$$A_2 = S_1^{-1} A_1 S_1 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix},$$

as the representation of φ_1 with respect to the new basis B_2. Our next step is to consider the endomorphism φ_2 of the subspace of $\mathbb{R}^3$ spanned by $(\mathbf{e}_2, \mathbf{e}_3)$ represented by the submatrix

$$A_2' = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

with respect to that basis. From the characteristic equation $p_{\varphi_2} = p_{A_2'} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$, we get $\lambda_2 = 1$. From the homogeneous system $(A_2 - E_2)\mathbf{v}_2 = 0$ we obtain

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

After extending $(\mathbf{v}_1, \mathbf{v}_2 = \mathbf{e}_2 - \mathbf{e}_3)$ to a new basis, for example: $B_3 = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3)$, we compute the transition matrix and its inverse:

$$S_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad S_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We derive

$$A_3 = S_2^{-1} A_1 S_2 = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

where A_3 has the desired upper triangle form.

Exercise T3 (Ideals)

Recall that a non-empty subset I of a commutative ring R is called an ideal, if it is closed under addition and under multiplication with arbitrary ring elements. The principal ideal I_a generated by a fixed element $a \in R$ is defined by

$$I_a = \{ra : r \in R\}$$

as the set of all multiples of a (see Definition 1.2.16 on page 22 of the notes).

(a) Verify that I_a is the smallest ($\subseteq$-minimal) ideal containing a.

(b) Let I and J be two ideals in a commutative ring R. Prove that

$$I + J = \{i + j : i \in I, j \in J\}$$

is again an ideal, in fact, the smallest ideal containing both I and J.

(c) Prove that every ideal over $\mathbb{Z}$ is principal. Is the same true in the rings $\mathbb{Z}_n$ $(n \in \mathbb{Z})$? (As already discussed in H3.3 from LA I 2010/11.)

(d) For two elements $m, n \in \mathbb{Z}$, the set $I_m + I_n$ is an ideal over $\mathbb{Z}$, hence principal. This means that $I_m + I_n = I_k$ for some element $k \in \mathbb{Z}$. Express k in terms of m and n.

(e) For any two ideals I and J in a commutative ring R, find an expression for $I \wedge J$, the largest ideal contained in both I and J. Over the ring $\mathbb{Z}$, how does one determine for any pair $m, n \in \mathbb{Z}$ the $k \in \mathbb{Z}$ such that $I_m \wedge I_n = I_k$?

Solution:

a) $a \in I_a$, $ra + sa = (r+s)a$ and $s(ra) = (sr)a$, so I_a is an ideal. Since ideals have to be closed under multiplication by arbitrary ring elements, $I_a \subseteq I$ for any ideal I containing a.

b) $I + J$ is non-empty, since both I and J are, and therefore $I + J$ is an ideal by the equalities $(i + j) + (i' + j') = (i + i') + (j + j')$ and $s(i + j) = si + sj$. Since ideals are closed under addition, $I + J \subseteq K$ for any ideal K containing both I and J.

c) Let I be a ideal over $\mathbb{Z}$ containing elements other than 0. Since $i \in I$ implies $-i \in I$, let a be the least positive element of I, and let $j \in I$. By the division algorithm, there exist integers k, l such that $j = ka + l$ with $0 \leq l < a$. Since $l = j - ka \in I$ and by definition of a, we have $l = 0$.

   Also for every ideal I in $\mathbb{Z}_n$, define

$$J := \{k \in \mathbb{Z} : k \bmod n \in I\}.$$

   J is easily seen to be an ideal in $\mathbb{Z}$, so it is generated by some $a \in \mathbb{Z}$. Therefore I is generated by $a \bmod n$.

d) From a previous OWO lecture or Exercise H3.3 from LA I 2010/11, it should be known that $I_m + I_n = \{am + bn : a, b \in \mathbb{Z}\}$ is precisely the set of all multiples of the greatest common divisor of m and n.

e) For any two ideals I and J in a commutative ring, their intersection $I \cap J$ is again an ideal, which is then clearly the largest contained in both.

   For $m, n \in \mathbb{Z}$, let I_m and I_n be the corresponding ideals. Recall that the least common multiple $\mathrm{lcm}(m, n)$ is an integer k characterized by the following properties:

   1. $m|k$ and $n|k$.
   2. If a is any integer for which $m|a$ and $n|a$, then $k|a$.

   We claim that $I_m \wedge I_n = I_k$, where $k = \mathrm{lcm}(m, n)$. Clearly $I_m \wedge I_n = I_k$ for some k, and we have $m|k$ and $n|k$ since $I_k \subseteq I_m$ and $I_k \subseteq I_n$. Suppose that $n|a$ and $m|a$ for some integer a. Then $a \in I_n \cap I_m$, so $a \in I_k$ by definition, and hence $k|a$. It follows that $k = \mathrm{lcm}(m, n)$.