Linear Algebra II
Tutorial Sheet no. 3

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Exercise T1 (Polynomials and polynomial functions)

Let $\mathbb{F}$ be a field (possibly finite). Recall that $\mathbb{F}[X]$ denotes the ring of polynomials

$$p = \sum_{i=0}^{n} a_i X^i, \quad a_i \in \mathbb{F} \quad n \geq 0.$$

For each element $p = \sum_{i=0}^{n} a_i X^i \in \mathbb{F}[X]$, we obtain a function $\check{p} : \mathbb{F} \to \mathbb{F}$, defined by $\check{p}(\lambda) := \sum_{i=0}^{n} a_i \lambda^i$ for $\lambda \in \mathbb{F}$. Recall that $\mathrm{Pol}(\mathbb{F})$ denotes the set of functions obtained in this way, which we call *polynomial functions*. Let $\check{}: \mathbb{F}[X] \to \mathrm{Pol}(\mathbb{F})$ denote the (surjective) map sending $p \mapsto \check{p}$. (You may refer, *e.g.*, to Exercise 1.2.1. in the lecture notes.)

(a) Show that $\check{}$ is a ring homomorphism. In other words, $\check{0} = 0$ and $\check{1} = 1$, where 0 (resp. 1) may also represent the constant function sending everything to 0 (resp. 1). Also, given $p, q \in \mathbb{F}[X]$, we have $(p+q)\check{} = \check{p} + \check{q}$ and $(pq)\check{} = \check{p}\check{q}$.

(b) Show that $\mathbb{F}[X]$ is infinite-dimensional for any $\mathbb{F}$ by showing that all elements X^k for $k \geq 0$ are linearly independent.

(c) Show that $\mathrm{Pol}(\mathbb{F})$ is finite-dimensional when $\mathbb{F} = \mathbb{F}_p$ for a prime p. Conclude that $\check{}$ is not an isomorphism in this case.

(d) Can you give an upper bound of the dimension of $\mathrm{Pol}(\mathbb{F})$? A better upper bound?

(e) Can you think of any other evaluation maps $\mathbb{F}[X] \to R$ for other rings R, defined in a similar way?

Solution:

a) Straightforward.

b) $\sum_{i=0}^{n} a_i X^i = 0 \iff \forall i,\, a_i = 0$.

c) Since $\mathbb{F}$ is finite, so is $\mathbb{F}^{\mathbb{F}}$, and so is $\mathrm{Pol}(\mathbb{F})$ since $\mathrm{Pol}(\mathbb{F}) \subseteq \mathbb{F}^{\mathbb{F}}$. Therefore the dimension of $\mathrm{Pol}(\mathbb{F})$ is also finite.

d) There are p^p elements in $\mathbb{F}^{\mathbb{F}}$, that is an upper bound for the dimension of $\mathrm{Pol}(\mathbb{F})$. Also $\lambda^p = \lambda$ for every $\lambda \in \mathbb{F}_p$. (See the little Fermat exercise, T4.2 of LA I in 2010/11.) It follows that $\check{X}^{i+p-1} = \check{X}^i$ for all $0 \leq i$. So $(\check{1}, \check{X}, \check{X}^2, \ldots, \check{X}^{p-2})$ spans $\mathrm{Pol}(\mathbb{F})$, so the dimension of $\mathrm{Pol}(\mathbb{F})$ is less than or equal to $p-1$.

e) Let A be an $n \times n$ matrix and R be the ring of matrices $n \times n$ over $\mathbb{F}$. Let $\tilde{}: \mathbb{F}[X] \to R$ such that $\tilde{p} := \sum_{i=0}^{n} a_i A^i$ for $p = \sum_{i=0}^{n} a_i X^i$. Notice that $a_0 A^0 = a_0 E$ where E denotes the identity matrix.

Exercise T2 (The Cayley-Hamilton Theorem for diagonalisable matrices)

(a) Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Recall from Exercise Sheet 2 that the characteristic polynomial $p_A(\lambda) = \lambda^2 - 5\lambda + 4$. Show that the characteristic polynomial of A annihilates A in the sense that $p_A(A) = A^2 - 5A + 4E = 0$, where E denotes the identity matrix and 0 denotes the zero matrix.

(b) Let $D = \mathrm{diag}(d_1, \ldots, d_n)$ be a diagonal matrix. Write down the characteristic polynomial $p_d(\lambda)$. Show that D satisfies its characteristic polynomial in the sense that $p_D(D) = O$ where O denotes the zero $n \times n$ matrix.

(c) Let A be a diagonalisable matrix. Without appeal to the Cayley-Hamilton Theorem (which will later generalise this assertion to all matrices), show that $p_A(A) = O$.

Solution:

a) Straightforward computation.

b) We have $p_D(\lambda) = (d_1 - \lambda) \cdots (d_n - \lambda)$. Substituting D for λ and $d_i E$ for d_i, where E is the identity matrix yields $p_D(D) = (d_1 E - D) \cdots (d_n E - D)$. Note that $d_j E - D$ is a diagonal matrix whose jth entry is zero. It follows that the product of these matrices is the zero matrix.

c) Let C be an invertible matrix and D be a diagonal matrix for which $A = CDC^{-1}$. Since the conjugation map given by $M \to CMC^{-1}$ is a ring homomorphism, it follows that

$$p_A(A) = p_A(CDC^{-1}) = Cp_A(D)C^{-1} = Cp_D(D)C^{-1}.$$

By part (a), $p_D(D) = O$, so we obtain $p_A(A) = O$ as well.

Exercise T3 (Polynomial division with remainder)

Let $p_1(x) = x^6 + 5x^5 + 6x^4 - x^3 + 2x^2 + 5x + 4$ and $p_2(x) = x^2 + 2x$. Using polynomial division with remainder, find the unique polynomials $q(x)$ and $r(x)$ with $\deg(r) < 2 = \deg(p_2)$ such that

$$p_1(x) = q(x)p_2(x) + r(x).$$

(See Definition 1.2.8. and Lemma 1.2.9 in the lecture notes.)

Solution:
$q(x) = x^4 + 3x^3 - x + 4$ and $r(x) = -3x + 4$.