Linear Algebra II
Tutorial Sheet no. 2

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Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise T1 (Geometric characterisation of linear maps by eigenvalues)

Give a geometric description of all the endomorphisms of $\mathbb{R}^3$ with the following sets of eigenvalues:

(a) $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1$

(b) $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

(c) $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$

Note that you cannot assume anything about the corresponding eigenvectors other than that they form a basis (why?).

Solution:

We denote the respective eigenvectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

a) $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1$: Since $\lambda_2 = 0$ all vectors are projected onto the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_3$. Within that plane, the vectors along $\mathbf{v}_3$ stay fixed, the ones along $\mathbf{v}_1$ are inverted. Thus we have a reflection. So the map describes a (skew) projection, followed by a (skew) reflection.

b) $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$: This map rescales vectors by factors 2 and 3 in directions $\mathbf{v}_2$ and $\mathbf{v}_3$, respectively.

c) $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$: This map sort of reflects the elements w.r.t the plane spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$ along $\mathbf{v}_1$. However, every vector is aditionally streched along $\mathbf{v}_3$. You might want to call this map a (skew) reflection w.r.t. the plane spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$ followed by a rescaling with factor 2 in the direction of $\mathbf{v}_3$.

Exercise T2 (Eigenvalues and eigenvectors over $\mathbb{R}$ and $\mathbb{C}$)

Let A be the 3×3-matrix $\begin{pmatrix} 0 & -1 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Determine the characteristic polynomial of the matrix A.

(b) Find all real eigenvalues of A and the corresponding eigenvectors of the map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with $\varphi(x) = Ax$.

(c) Find all eigenvalues for the corresponding map $\varphi : \mathbb{C}^3 \to \mathbb{C}^3$ with $\varphi(x) = Ax$ and give a basis of each eigenspace.

Solution:

a) $p_A(\lambda) = \det \begin{pmatrix} -\lambda & -1 & 4 \\ 1 & -\lambda & 2 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda) \cdot \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = (1-\lambda)(\lambda^2+1).$

b)

$$p_A(\lambda) = 0 \Leftrightarrow 1 - \lambda = 0 \text{ or } \lambda^2 + 1 = 0$$

so there is only one real eigenvalue $\lambda_1 = 1$ and the corresponding eigenspace is span $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \{r \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} : r \in \mathbb{R}\}$

c)

$$p_A(\lambda) = (1-\lambda)(\lambda^2+1) = 0 \Leftrightarrow 1 - \lambda = 0 \text{ or } \lambda + i = 0 \text{ or } \lambda - i = 0$$

So the eigenspaces are

$$\lambda_1 = \ \ 1\colon\ \{r\begin{pmatrix}1\\3\\1\end{pmatrix} : r\in\mathbb{C}\} \text{ with basis } b_1 = \begin{pmatrix}1\\3\\1\end{pmatrix}$$

$$\lambda_2 = \ \ i\colon\ \{r\begin{pmatrix}i\\1\\0\end{pmatrix} : r\in\mathbb{C}\} \text{ with basis } b_1 = \begin{pmatrix}i\\1\\0\end{pmatrix}$$

$$\lambda_3 = -i\colon\ \{r\begin{pmatrix}1\\i\\0\end{pmatrix} : r\in\mathbb{C}\} \text{ with basis } b_1 = \begin{pmatrix}1\\i\\0\end{pmatrix}$$

Exercise T3 (Diagonalisation)

Consider the matrix $A = \begin{pmatrix}2 & 2\\1 & 3\end{pmatrix}$ over $\mathbb{R}$.

(a) Determine all eigenvalues of A and corresponding eigenvectors.

(b) Find a regular matrix C such that $D = C^{-1}AC$ is a diagonal matrix.

(c) Calculate A^6.

(d) Find a "positive square root" of A, i.e., find a matrix R with non-negative eigenvalues such that $R^2 = A$

(e) Check that $t \mapsto e^{tA}\mathbf{v}_0$ solves the differential equation $\frac{d}{dt}\mathbf{v}(t) = A\mathbf{v}(t)$ with initial value $\mathbf{v}(0) = \mathbf{v}_0$.

Solution:

a) We have

$$\det(A-\lambda E) = \lambda^2 - 5\lambda + 4 = (\lambda-4)(\lambda-1).$$

Thus the eigenvalues of φ are $\lambda_1 = 4$ and $\lambda_2 = 1$. In order to determine the kernels of $A - \lambda_i E$, we perform Gauß-Jordan elimination:

$$\begin{aligned} A-\lambda_1 E &= \begin{pmatrix}-2 & 2\\1 & -1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\\ A-\lambda_2 E &= \begin{pmatrix}1 & 2\\1 & 2\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 2\\0 & 0\end{pmatrix}. \end{aligned}$$

We may choose $\mathbf{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ with $\mathrm{span}(\mathbf{v}_1) = \ker(A-\lambda_1 E)$ and $\mathbf{v}_2 = \begin{pmatrix}-2\\1\end{pmatrix}$ with $\mathrm{span}(\mathbf{v}_2) = \ker(A-\lambda_2 E)$.

b) Let $\varphi = \varphi_A$ be the linear map represented by A w.r.t. the standard basis. Then φ is represented by the diagonal matrix $D = \begin{pmatrix}4 & 0\\0 & 1\end{pmatrix}$ w.r.t. the (labelled) basis $B = (\mathbf{v}_1, \mathbf{v}_2)$ of $\mathbb{R}^2$.

Therefore the desired matrix C is the transition matrix from the basis B to the standard basis, i.e. $C := \begin{pmatrix}1 & -2\\1 & 1\end{pmatrix}$.

It is easy to verify that its inverse is $C^{-1} = \frac{1}{3}\begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix}$, so that $D = C^{-1}AC = \begin{pmatrix}4 & 0\\0 & 1\end{pmatrix}$ is indeed a diagonal matrix.

c) The conjugation map $\rho_C : \mathbb{F}^{2,2} \to \mathbb{F}^{2,2}$ given by $M \mapsto C^{-1}MC$ is an automorphism of the ring $\mathbb{F}^{2,2}$ of 2×2 matrices, and in particular preserves sums and products of matrices. Since $\rho_c(A) = D$ we have $\rho_c(A^k) = D^k$ for all $0 \le k$. Similarly $A^k = \rho_C^{-1}(D^k) = (CDC^{-1})^k = CD^kC^{-1}$.

$$\begin{aligned} A^6 &= (CDC^{-1})^6 = CD^6C^{-1} = \tfrac{1}{3}\begin{pmatrix}1 & -2\\1 & 1\end{pmatrix}\begin{pmatrix}4^6 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix}\\ &= \tfrac{1}{3}\begin{pmatrix}4^6 & -2\\4^6 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix} = \tfrac{1}{3}\begin{pmatrix}4^6+2 & 2\cdot 4^6-2\\4^6-1 & 2\cdot 4^6+1\end{pmatrix} = \begin{pmatrix}1366 & 2730\\1365 & 2731\end{pmatrix}. \end{aligned}$$

d) Take $R = C\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}C^{-1}$. Since

$$R^2 = C\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}C^{-1}C\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}C^{-1} = CDC^{-1} = A,$$

R is a "positive square root" of A.

e) Recall that the exponential function over the reals is defined by $e^x = \sum_{0 \le k} \frac{x^k}{k!}$ for all x in $\mathbb{R}$. Similarly for a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$, since $D^k = \operatorname{diag}(d_1^k, \dots, d_n^k)$, it is natural to define

$$e^D = \sum_{k=0}^{\infty} \frac{D^k}{k!} = \operatorname{diag}(e^{d_1}, \dots, e^{d_n}).$$

Next, suppose that A is diagonalisable and write $A = CDC^{-1}$ where D is diagonal. Since $A^k = CD^kC^{-1}$ for all k, we have

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{CD^kC^{-1}}{k!} = C\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right)C^{-1} = Ce^DC^{-1}.$$

Therefore $e^{tA} = Ce^{tD}C^{-1}$. Note also that $e^{tD} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}$, so that

$$\frac{d}{dt}e^{tD} = \frac{d}{dt}\begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} 4e^{4t} & 0 \\ 0 & e^t \end{pmatrix} = D\begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix} = De^{tD}.$$

Therefore:

$$\begin{aligned} \frac{d}{dt}\mathbf{v}(t) &= \frac{d}{dt}e^{tA}\mathbf{v}_0 = \frac{d}{dt}\left(Ce^{tD}C^{-1}\right)\mathbf{v}_0 = C\left(\frac{d}{dt}e^{tD}\right)C^{-1}\mathbf{v}_0 \\ &= CDe^{tD}C^{-1}\mathbf{v}_0 = \left(CDC^{-1}\right)\left(Ce^{tD}C^{-1}\right)\mathbf{v}_0 = Ae^{tA}\mathbf{v}_0 = A\mathbf{v}(t). \end{aligned}$$

Exercise T4 (Eigenvalues of nilpotent maps)

Let V be a vector space of dimension greater than 0, and let $\varphi : V \to V$ be a nilpotent endomorphism, that is, an endomorphism such that $\varphi^k = \mathbf{0}$ for some $k \in \mathbb{N}$.

(a) Show that 0 is the only possible eigenvalue of φ.

(b) Show that 0 is an eigenvalue of φ.

Solution:

a) If λ is some eigenvalue of φ, then $\varphi(\mathbf{v}) = \lambda\mathbf{v}$ for some non-null vector $\mathbf{v} \in V$. Then $\mathbf{0} = \varphi^k(\mathbf{v}) = \lambda^k\mathbf{v}$. Since $\mathbf{v}$ was non-null, this implies that $\lambda^k = 0$ and therefore that $\lambda = 0$.

b) Note that $\mathbf{0}$ is an eigenvalue of φ if and only if $\ker(\varphi)$ is nontrivial. But $\ker(\varphi) = \mathbf{0}$ implies that φ is regular, which implies that φ^k is regular for all $k \ge 0$. This contradicts the fact that φ is nilpotent.