Linear Algebra II
Tutorial Sheet no. 1

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Summer term 2011
April 13, 2011

Discuss and compare as many different solution strategies as possible for the following two questions from your exam.

Exercise T1 (Exam problem 2)

Let $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ be an ordered basis of an n-dimensional $\mathbb{F}$-vector space V.

(a) Let B' be obtained by replacing $\mathbf{b}_i$ by $\mathbf{b}'_i = \sum_{j=1}^{i} \mathbf{b}_j$ for $1 \le i \le n$:

$$B' := (\mathbf{b}_1,\ \mathbf{b}_1 + \mathbf{b}_2,\ \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3,\ \dots\ , \mathbf{b}_1 + \cdots + \mathbf{b}_n).$$

Determine whether B' always also forms a basis of V.

(b) For $\mathbf{v} \in V$ let

$$B - \mathbf{v} := (\mathbf{b}_1 - \mathbf{v},\ \mathbf{b}_2 - \mathbf{v},\ \dots\ ,\ \mathbf{b}_n - \mathbf{v}).$$

Show that the set of those $\mathbf{v} \in V$ for which $B - \mathbf{v}$ is *not* a basis of V forms an affine subspace of dimension $n-1$ (which contains, and is therefore spanned by, the $\mathbf{b}_i$).

Hint: turn the condition that $B - \mathbf{v}$ admits a non-trivial linear combination of $\mathbf{0}$ into a vector equation for $\mathbf{v}$.

Solution:

a) Let $\varphi : V \to V$ be the map $\varphi(\mathbf{b}_i) = \mathbf{b}'_i$. It is easy to check that the matrix $[\![\varphi]\!]_B^B$ of φ has 1's on and above the diagonal, and zeroes below the diagonal. The determinant of this matrix is 1, so φ is invertible and B' also forms a basis of V.

b) Suppose that $B - \mathbf{v}$ is not a basis of V. Then there are constants $\lambda_1, \dots, \lambda_n$, not all zero, such that

$$\lambda_1(\mathbf{b}_1 - \mathbf{v}) + \cdots + \lambda_n(\mathbf{b}_n - \mathbf{v}) = 0.$$

Then

$$\lambda_1 \mathbf{b}_1 + \cdots + \lambda_n \mathbf{b}_n = \lambda \mathbf{v}$$

where $\lambda = \sum_{i=1}^{n} \lambda_n$. First, we claim that $\lambda \neq 0$; otherwise we would have a relation of linear dependence among the elements of B. Hence we can divide both sides by λ, obtaining

$$\mu_1 \mathbf{b}_1 + \cdots + \mu_n \mathbf{b}_n = \mathbf{v}$$

where $\mu_i = \frac{\lambda_i}{\lambda}$. Clearly $\sum_{i=1}^{n} \mu_i = 1$, and this is the only condition on $\mathbf{v}$. Hence the set of all $\mathbf{v}$ such that $B - \mathbf{v}$ is not a basis of V is precisely the set of linear combinations $\mu_1 \mathbf{b}_1 + \cdots + \mu_n \mathbf{b}_n$ such that $\sum_{i=1}^{n} \mu_i = 1$. This is an affine subset of dimension $n-1$.

Exercise T2 (Exam Problem 4)

In $V = \mathbb{R}^4$, let $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map with

$$\varphi((1,0,0,1)) = (2,0,0,1), \qquad \varphi((2,0,0,1)) = (0,1,1,0),$$
$$\varphi((0,1,1,0)) = (0,1,2,0), \qquad \varphi((0,1,2,0)) = (1,0,0,1).$$

(a) Check that $\mathbf{b}_1 = (1,0,0,1)$, $\mathbf{b}_2 = (2,0,0,1)$, $\mathbf{b}_3 = (0,1,1,0)$, $\mathbf{b}_4 = (0,1,2,0)$ form a basis $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$ of $\mathbb{R}^4$ and determine the matrix representation $[\![\varphi]\!]_B^B$ of φ.
Is φ injective? Does it have an inverse?

(b) Let $S = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ be the standard basis. Derive the matrix representations $[\![\varphi]\!]_S^B$ and $[\![\varphi]\!]_S^S$ from $[\![\varphi]\!]_B^B$ through a systematic application of suitable basis transformation matrices.

Solution:

a) It is easy to check that $[\![\varphi]\!]_B^B$ has the form $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. This matrix is invertible (being a permutation matrix), and its inverse is just the transpose $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

b) By definition, $[\![\varphi]\!]_S^B$ is given by $\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Similarly, $[\![\mathrm{id}]\!]_S^B$ is given by $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \end{pmatrix}$. Since $[\![\mathrm{id}]\!]_B^S [\![\mathrm{id}]\!]_S^B = [\![\mathrm{id}]\!]_S^S$, it follows that $[\![\mathrm{id}]\!]_B^S$ is the inverse of $[\![\mathrm{id}]\!]_S^B$. By an easy computation, $[\![\mathrm{id}]\!]_B^S$ is given by $\begin{pmatrix} -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$. It follows that

$$[\![\varphi]\!]_S^S = [\![\varphi]\!]_S^B [\![\mathrm{id}]\!]_B^S = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 1 & 4 \\ 1 & 2 & -1 & -1 \\ 1 & 4 & -2 & -1 \\ -1 & -1 & 1 & 2 \end{pmatrix}.$$

Exercise T3 (Complex numbers)

Recall that complex numbers are represented by expressions of the form

$$z = a + bi$$

with $a, b \in \mathbb{R}$, $i \notin \mathbb{R}$ a new constant. Identifying $a \in \mathbb{R}$ with the complex number $a + 0i$ and the new constant i with $0 + 1i$, one may introduce addition and multiplication as the natural extensions of addition and multiplication in $\mathbb{R}$ based on associativity, commutativity, distributivity and the identity $i^2 = -1$. $\mathbb{R}$ thus becomes a subfield of the field of complex numbers.

(a) Let $z_1 = 3 + 4i$ and $z_2 = 5 + 12i$ be complex numbers. Compute

$$z_1^{-1}, \quad z_2^{-1}, \quad z_1^2, \quad z_2^2, \quad \text{and} \quad z_1 z_2,$$

and draw them in the complex plane. Find the complex square roots of i, z_1 and z_2, i.e., solve the equations $x^2 = i, x^2 = z_1, x^2 = z_2$ over $\mathbb{C}$.

(b) Define for $\varphi \in \mathbb{R}$,

$$e^{i\varphi} := \cos\varphi + i \sin\varphi.$$

Show that $e^{i\varphi} e^{i\psi} = e^{i(\varphi+\psi)}$ and $(e^{i\varphi})^n = e^{in\varphi}$ for every natural number n.

(c) Show that every complex number $z \in \mathbb{C} \setminus \{0\}$ can be represented as:

$$z = re^{i\varphi},$$

with $r \in \mathbb{R}_{>0}$. Prove that this representation is unique in the following sense:
$z = se^{i\psi}$ with $s > 0$ implies $r = s$ and $\varphi \equiv \psi \bmod 2\pi$.

(d) Use the representation from (c) to

i. give a geometric description of complex multiplication in terms of rotations and rescalings (i.e., dilations or contractions) in $\mathbb{R}^2$.

ii. find all complex solutions of $z^5 = 1$ and draw these in the complex plane. In general, find all solutions to $z^n = w$ for $w \in \mathbb{C}\backslash\{0\}, n \in \mathbb{N}$.

Solution:

a)

$$z_1^{-1} = \frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{3-4i}{25}, \quad z_2^{-1} = \frac{1}{5+12i} = \frac{5-12i}{169}$$

$$z_1^2 = -7+24i \quad z_2^2 = -119+120i, \quad \text{and } z_1 z_2 = -33+56i$$

For $x^2 = i$:

$$x_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \text{ and } x_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i.$$

For $x^2 = z_1$:

$$x_1 = 2+i \text{ and } x_2 = -2-i.$$

For $x^2 = z_2$:

$$x_1 = 3+2i \text{ and } x_2 = -3-2i.$$

b)

$$\begin{aligned} e^{i\varphi} e^{i\psi} &= (\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi) \\ &= (\cos\varphi\cos\psi - \sin\varphi\sin\psi) + (\cos\varphi\sin\psi + \sin\varphi\cos\psi)i \\ &= \cos(\varphi+\psi) + \sin(\varphi+\psi)i \\ &= e^{i(\varphi+\psi)}, \end{aligned}$$

using the trigonometric formulas for $\cos(\varphi+\psi)$ and $\sin(\varphi+\psi)$.

The equality $(e^{i\varphi})^n = e^{in\varphi}$ then follows by induction on n.

c) Let $z = a+bi \neq 0$. Trying to find a representation $z = re^{i\varphi} = r\cos\varphi + (r\sin\varphi)i$ means solving $a = r\cos\varphi$ and $b = r\sin\varphi$. One finds r by observing that

$$a^2 + b^2 = r^2(\cos^2\varphi + \sin^2\varphi) = r^2,$$

so $r = \sqrt{a^2+b^2} > 0$ and r is uniquely determined (it is the *modulus* of z). Furthermore, φ has to be an angle such that

$$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}.$$

This has a unique solution $\varphi_0 \in [0, 2\pi)$, called the *argument* of z. The argument of z is just the angle between the positive part of the x-axis and the vector z in the complex plane. Furthermore, the set of all solutions is given by $\{\varphi_0 + 2k\pi : k \in \mathbb{Z}\}$.

d)

1. Let $z \in \mathbb{C}$. If $z = 0$, then $wz = 0$ for all $w \in \mathbb{C}$. Assume that $z \neq 0$, so by (iii), it has the form $z = re^{i\varphi}$ with $r > 0$ and $\varphi \in [0, 2\pi)$. Then, for any complex number w, multiplication of w by z is equivalent to a rotation of w through angle φ followed by a rescaling using the modulus r of z.
2. $z = 0$ is certainly not a solution of $z^5 = 1$, so assume z is of the form $re^{i\varphi}$ with $r > 0$. We have to solve the equation:

   $$(re^{i\varphi})^5 = 1, \quad \text{that is} \quad r^5 e^{i5\varphi} = 1e^{i0}.$$

   By the uniqueness of the representation, this implies $r^5 = 1$ and $5\varphi = 0 \bmod 2\pi$. So $r = 1$, since $r > 0$. Then by solving $5\varphi = 2\pi k$ for every integer k with $0 \leq k < 5$, we find solutions $\varphi_k = \frac{2\pi k}{5} \in [0, 2\pi)$. This means we have found five different solutions (viz, $e^{i\varphi_k}$ with $0 \leq k < 5$), which must be all, since a fifth degree equations can have at most five solutions.

   For general $w = e^{i\varphi} \in \mathbb{C} \setminus \{0\}$, the equation $z^n = re^{i\varphi}$ has n solutions

   $$z_k = \sqrt[n]{r} e^{i\varphi_k}, \text{ with } \varphi_k = \frac{2\pi k}{n}, \quad k = 0, \ldots, n-1.$$