Linear Algebra II
Exercise Sheet no. 15

Summer term 2011
July 13, 2011

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Exercise 1 (Characteristic and minimal polynomials)

Find the characteristic and minimal polynomials of the following matrix

$$A = \begin{pmatrix} 1 & 1 & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & 1 & & & \text{0} & & & \\ & & & 1 & & & & & & \\ & & & & 2 & 1 & & & & \\ & & & & & 2 & 1 & & & \\ & & & & & & 2 & & & \\ & & & \text{0} & & & & 2 & & \\ & & & & & & & & 4 & \\ & & & & & & & & & 4 \end{pmatrix}$$

Exercise 2 (Jordan normal form)

(a) Let φ be an endomorphism of a ten-dimensional $\mathbb{F}$-vector space V. W.r.t. basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_{10})$ let φ be represented by a Jordan normal form matrix with three Jordan blocks for the same eigenvalue $\lambda \in \mathbb{F}$, of sizes 2, 3 and 5. Let $\psi := \varphi - \lambda\text{id}$. Complete the following table:

i	1	2	3	4	5	6	7	8	9	10
$\dim(\ker \psi^i)$					10	10	10	10	10	10

In the notation of Lemma 1.6.4 of the notes: for which $\mathbf{v} \in V$ does $[\![\mathbf{v}]\!]$ have maximal dimension? Split the basis B in a way to obtain bases for the two invariant subspaces $V = [\![\mathbf{v}]\!] \oplus V'$ (as in Claim 1.6.5). If φ' is the restriction of φ to V', what is the matrix representation of φ' with respect to this basis? If $\psi' = \varphi' - \lambda\text{id}$, how is the corresponding table for ψ' related to the above?

(b) Now, let φ be another endomorphism of V with characteristic polynomial $(\lambda - X)^{10}$. Suppose we have the following data for $\psi = \varphi - \lambda\text{id}$:

i	1	2	3	4	5	6	7	8	9	10
$\dim(\ker \psi^i)$	3	5	7	8	9	10	10	10	10	10

Determine the Jordan normal form representation of φ from this data (up to permutation of Jordan blocks).

(c) (extra) In general, let φ_0 and φ_1 be two endomorphisms of $\mathbb{F}$-vector spaces V_0 and V_1 of the same finite dimension, with the same characteristic polynomial that splits into linear factors. Suppose moreover that for each eigenvalue λ of φ_0 and φ_1, the tables for $\psi_0 = \varphi_0 - \lambda\text{id}$ and $\psi_1 = \varphi_1 - \lambda\text{id}$ are the same.

Sketch a proof for the similarity of φ_0 and φ_1 adapting the argument for the existence and uniqueness of the Jordan normal form. How can this be used to give a "different" proof for the similarity of A and A^t for any matrix $A \in \mathbb{C}^{(n,n)}$?

Exercise 3 (Diagonalization using orthogonal matrices)

Let φ be the endomorphism of $\mathbb{R}^3$ given in the standard basis by

$$A = \begin{pmatrix} 1 & -1 & -1/\sqrt{2} \\ -1 & 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 2 \end{pmatrix}.$$

(a) Find an orthogonal matrix B such that $B^{-1}AB$ is diagonal.

(b) Describe all subspaces of $\mathbb{R}^3$ which are invariant under φ.

(c) For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, let Q be the quadratic form

$$Q(\mathbf{x}) = x_1^2 + x_2^2 + 2x_3^2 - 2x_1x_2 - \frac{2}{\sqrt{2}}x_1x_3 + \frac{2}{\sqrt{2}}x_2x_3.$$

Find the principle axes of the quadric X given by $Q(\mathbf{x}) = 1$.

Exercise 4 (Invariant planes in $\mathbb{R}^4$)

Let φ be an orthogonal transformation of $\mathbb{R}^4$ which fixes a plane U_1 pointwise, and acts by a nontrivial rotation on another plane U_2. Prove that U_1 and U_2 are the only invariant subspaces of $\mathbb{R}^4$ of dimension 2.