Linear Algebra II
Exercise Sheet no. 14

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Exercise 1 (Bijection between conic sections)

Let $\mathbb{X}$ be the standard cone in $\mathbb{R}^3$ (defined by the equation $x_1^2 + x_2^2 = x_3^2$), let $\mathbb{A}_1, \mathbb{A}_2$ be the planes defined by the equations $x_3 = 1$ and $x_1 = \frac{1}{3}x_3 + \frac{2}{3}$, respectively and let L be the line that goes through the vectors $\begin{pmatrix} \frac{1}{3} \\ \pm\frac{2}{3} \\ 1 \end{pmatrix}$.

(a) Let $\mathbf{n} := \begin{pmatrix} -1 \\ 0 \\ \frac{1}{3} \end{pmatrix}$. Then the map

$$\varphi : \mathbb{A}_1 \setminus L \to \mathbb{A}_2$$
$$\mathbf{v} \mapsto -\frac{2}{3}\frac{1}{\langle \mathbf{n}, \mathbf{v} \rangle}\mathbf{v}$$

describes the central projection through the origin from $\mathbb{A}_1 \setminus L$ into $\mathbb{A}_2$. Make a sketch to verify this. (You only need to draw the (x_1, x_3)-plane.) Determine the image of φ.

(b) Sketch the conic sections that you get from $\mathbb{A}_i$ and $\mathbb{X}$, $i = 1, 2$. (You only need to draw the (x_1, x_3)-plane.)

(c) Compute a parametric description of $\mathbb{A}_1 \cap \mathbb{X}$ and $\mathbb{A}_2 \cap \mathbb{X}$.

(d) How can you extend φ to a bijection from the one conic section $(\mathbb{A}_1 \cap \mathbb{X})$ onto a completion of the other?

Exercise 2 (Minkowski space)

Consider the “Minkowski metric” on $\mathbb{R}^4$ induced by the symmetric bilinear form σ with diagonal entries $(1, 1, 1, -1)$ w.r.t. the standard basis.
The quadric $Q = \{\mathbf{v} \in \mathbb{R}^4 : \sigma(\mathbf{v}, \mathbf{v}) = 0\}$ is called the *null set* of σ.

(a) Show that σ is non-degenerate but has a non-trivial null set; determine the null set and describe it geometrically.

(b) Give examples of other bases of $\mathbb{R}^4$ w.r.t. which σ is represented by the matrix with diagonal entries $(1, 1, 1, -1)$, but which are not orthonormal w.r.t. the standard scalar product.

(c) Give an example of a subspace $U \subseteq \mathbb{R}^4$ s.t. $\mathbb{R}^4 \neq U \oplus U^\perp$ where

$$U^\perp := \{\mathbf{v} \in \mathbb{R}^4 : \sigma(\mathbf{v}, \mathbf{u}) = 0 \text{ for all } \mathbf{u} \in U\}$$

(d) Which are the signatures of the quadratic forms induced by σ on the 3-dimensional subspaces $U \subseteq \mathbb{R}^4$? Try to describe in each case the relation between the subspace U and the null set.

Exercise 3 (A rotated ellipse)

Let X be the ellipse in $\mathbb{R}^2$ obtained by rotating the standard ellipse $\frac{x^2}{4} + y^2 = 1$ through the angle $-\frac{\pi}{6}$ and translating it so that its center is at the point $(1, -1)$. Find an equation for X.

Exercise 4 (Projection onto a plane)

Let A be the affine plane in the euclidean space $(\mathbb{R}^3, \langle, \rangle)$ given by $x + 2y + 2z = 9$.

(a) Find an orthonormal basis for the 2-dimensional linear subspace $U \subseteq \mathbb{R}^3$ which is parallel to A.

(b) Extend this basis to an orthonormal basis B for $\mathbb{R}^3$.

(c) Write down the matrix which represents the orthogonal projection φ onto U in terms of the standard basis $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^3$.

(d) Let P be the point $(1, 2, -1)$. Find the shortest distance from P to A.