Linear Algebra II
Exercise Sheet no. 13

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Exercise 1 (Warm-up: symmetries of quadrics)
Consider the quadric

$$\mathbb{X} = \{\mathbf{v} \in \mathbb{R}^n : Q(\mathbf{v}) = c\},$$

where Q is a quadratic form over $\mathbb{R}^n$, $c \in \mathbb{R}$. Show that $\mathbb{X}$ is invariant under the following linear isometries of $\mathbb{R}^n$:

(a) $-\mathrm{id} : \mathbf{x} \mapsto -\mathbf{x}$ (central symmetry);

(b) reflection in the hyperplanes orthogonal to a principal axis (i.e., hyperplanes spanned by any $n-1$ basis vectors from an orthonormal basis that diagonalises Q and the associated σ);

(c) rotations in planes spanned by two principal axes w.r.t. which Q has the same "eigenvalues", i.e., by basis vectors $\mathbf{b}, \mathbf{b}'$ from an orthonormal basis that diagonalises Q such that $Q(\mathbf{b}) = Q(\mathbf{b}')$.

Exercise 2 (Canonical form of a quadric)
Let $\mathbb{X}$ be the set of all points $\mathbf{x} \in \mathbb{R}^3$ satisfying the following equation:

$$2x_1^2 + x_2^2 + x_3^2 + 2x_2x_3 + 4x_1 - 3x_2 - x_3 = 0.$$

(a) Find a matrix A and a vector $\mathbf{b}$ such that the given equation can be written as

$$\mathbf{x}^t A\mathbf{x} + \mathbf{b}^t\mathbf{x} = 0.$$

(b) Find an affine transformation for $\mathbb{R}^3$ for which the given equation has the form

$$a(x_1' + c_1)^2 + b(x_2' - c_2)^2 + x_3' - c_3 = 0.$$

(c) Describe the set $\mathbb{X}$ geometrically.

Exercise 3 (Three-dimensional quadrics)
Various three-dimensional affine quadrics as for instance the single-sheet hyperboloid $\{(x,y,z) : x^2 + y^2 - z^2 = 1\}$ and the saddle surface $\{(x,y,z) : x^2 - y^2 = z\}$, can be seen to be different affine sections of the projective quadric

$$\mathbb{X} = \{[\mathbf{x}] : x_1x_2 - x_3x_4 = 0\}. \qquad (*)$$

(a) Find the matrix representing $\mathbb{X}$ w.r.t. the standard basis.

(b) Diagonalise the quadratic form for $\mathbb{X}$.

(c) Use $(*)$ to find an affine hyperplane whose intersection with $\mathbb{X}$ is a saddle surface and use (b) to find a plane whose intersection with $\mathbb{X}$ is a single-sheet hyperboloid.

(d) Give a homogeneous equation for another projective quadric $\mathbb{X}'$ such that there exist two affine hyperplanes whose intersections with $\mathbb{X}'$ are a double-sheet hyperboloid and an ellipsoid, respectively.

Exercise 4 (Snake on a plane)

A snake wants to buy a blanket. The snake's length is one unit, and we assume that it can bend any which way, but may always be described by a smooth curve of length 1. The snake wants to make sure that it can cover itself with the blanket no matter in which shape it wants to lie down. Obviously a round blanket of diameter 1 is good enough (why?). A clever shop assistant points out that a half disc of diameter 1 should also suffice.

Prove that this is right: any length 1 curve in $\mathbb{R}^2$ can be covered by a half disc of diameter 1.

Hint: Consider the end points and the mid point of the curve, and use the fact that the whole of the snake lies within the union of the two ellipses formed by the mid point with either end point as foci and with length 1/2 for the sum of distances from the foci. It now suffices to show that any two such ellipses are contained within a half disc of radius 1/2 whose straight boundary is a common tangent to the two ellipses, and whose centre point is the orthogonal projection of the shared focus point onto this tangent. (See Exercise E12.5.)