Linear Algebra II
Exercise Sheet no. 12

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Exercise 1 (Warm-up: diagonalisability of bilinear forms)
Let the bilinear forms σ_1 and σ_2 on $\mathbb{R}^3$ be defined by the matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

with respect to the standard basis of $\mathbb{R}^3$.

(a) Determine an orthonormal basis of $\mathbb{R}^3$ with respect to which the matrix of σ_1 is diagonal.

(b) Show that every eigenvector of A_2 is also an eigenvector of A_1.

(c) Determine an orthonormal basis of $\mathbb{R}^3$ with respect to which the matrix of σ_2 is diagonal, and deduce the eigenvalues of A_2 without computing the characteristic polynomial.

Exercise 2 (Quadratic forms)
Which of the following are quadratic forms? Determine in each case the corresponding symmetric bilinear forms:

(a) $Q_1 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x.$

(b) $Q_2 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 0.$

(c) $Q_3 : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto -3x_1^2 - x_2^2 - x_2 x_1.$

(d) $Q_4 : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto (\sqrt{x_1} + \sqrt{x_2})^4.$

(e) $Q_5 : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto ||\mathbf{x}||^2.$

Exercise 3 (Transformation of quadratic forms)
Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the endomorphism represented w.r.t. the standard basis of $\mathbb{R}^2$ by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$\mathbb{S}^1 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x}^t \mathbf{x} = 1\}.$$

be the unit circle in $\mathbb{R}^2$.

(a) Describe the image of the unit circle $\mathbb{S}^1$ under φ, $\varphi[\mathbb{S}^1] \subseteq \mathbb{R}^2$, by a corresponding equation.

(b) Determine a symmetric bilinear form σ such that

$$\varphi[\mathbb{S}^1] = \{\mathbf{x} \in \mathbb{R}^2 : \sigma(\mathbf{x}, \mathbf{x}) = 1\}.$$

(c) Find the symmetry axes of $\varphi[\mathbb{S}^1]$.
Hint: apply Theorem 3.2.5 to σ.

Exercise 4 (Quadratic forms)
Determine the principal axes and the signature of the following quadratic forms

(a) $Q_1(\mathbf{x}) = -11x_1^2 - 16x_1x_2 + x_2^2$,

(b) $Q_2(\mathbf{x}) = 9x_1^2 - 4x_1x_2 + 6x_2^2$,

(c) $Q_3(\mathbf{x}) = 4x_1^2 - 12x_1x_2 + 9x_2^2$,

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

Exercise 5 (Geometric properties of the ellipse in euclidean geometry)

Let $0 \leqslant e < 1$ and consider the points given by the vectors $\mathbf{e} = (e, 0)$ and $-\mathbf{e} = (-e, 0)$ in the real plane $\mathbb{R}^2$. Let the set $\mathbb{X}_e \subseteq \mathbb{R}^2$ be defined as the set of all those $\mathbf{x} \in \mathbb{R}^2$ for which $d(\mathbf{x}, \mathbf{e}) + d(\mathbf{x}, -\mathbf{e}) = 2$.

(a) Show that $\mathbb{X}_e$ is an *ellipse* defined by a quadratic equation of the form $\alpha x^2 + \beta y^2 = 1$ for suitable $\alpha, \beta > 0$. Determine α and β in terms of e. Draw $\mathbb{X}_e$ for $e = 0, 1/2, 1, 1/\sqrt{2}$.

(b) From (i) find a representation of $\mathbb{X}_e$ as the image of the unit circle under a rescaling in the y-direction. Use this rescaling and the fact that linear transformations preserve the property that a line is a tangent to a curve in order to determine the equation of the tangent to the ellipse $\mathbb{X}_e$ in a point $\mathbf{x} = (x, y) \in \mathbb{X}_e$. Show that lines from $\mathbf{e}$ and $-\mathbf{e}$ through $\mathbf{x}$ form the same angle with the tangent at $\mathbf{x}$. [This explains the rôle of the points $\mathbf{e}$ and $-\mathbf{e}$ as the *foci* of the ellipse: light shining from $\mathbf{e}$ is focussed in $-\mathbf{e}$ after reflection in $\mathbb{X}_e$.]

(c) Show by elementary geometric means that $\mathbb{X}_e$ also has the following geometric property. Let t be the tangent to $\mathbb{X}_e$ in a point $\mathbf{x} \in \mathbb{X}_e$ and l the line through $\mathbf{e}$ perpendicular to t. Then the point of intersection $\mathbf{v}$ between l and t lies on the unit circle.

Hint: Consider the triangles $(\mathbf{x}, \mathbf{v}, \mathbf{w})$ and $(\mathbf{x}, \mathbf{v}, \mathbf{e})$ in the sketch below, where $\mathbf{v}$ marks the point where l intersects t, and $\mathbf{w}$ where it intersects the line through $-\mathbf{e}$ and $\mathbf{x}$. Use (ii) to argue that these triangles are congruent.