Linear Algebra II
Exercise Sheet no. 11

TECHNISCHE UNIVERSITÄT DARMSTADT

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Summer term 2011
June 20, 2011

Exercise 1 (Warm-up)

(Exercise 3.1.1 in the notes, see also T7.2.) Show that the relation $\approx$ on $\mathbb{R}^{(n,n)}$ defined as $A \approx A'$ iff $A' = C^t AC$ for some $C \in \mathrm{GL}_n(\mathbb{R})$ is an equivalence relation. What are sufficient criteria for $A \not\approx A'$?

Exercise 2 (Normal matrices)

Recall that a matrix A is called normal if $AA^+ = A^+A$. We have seen (cf Exercise T11.1) that unitary, hermitian, and skew-hermitian matrices are normal. (Similarly in the real case, orthogonal, symmetric, skew-symmetric matrices are normal.) In this exercise we will see that there are normal matrices that do not belong to any of these classes.

(a) Prove that every real 2×2 normal matrix is either symmetric or a scalar multiple of an orthogonal matrix.

(b) Find a sufficient (and also necessary) condition for a complex 2×2 matrix to be normal. Give an example of such a matrix which is neither hermitian, skew-hermitian, nor a scalar multiple of a unitary matrix.

(c) Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Show that A is normal, but is neither symmetric, skew-symmetric, nor a scalar multiple of an orthogonal matrix.

Exercise 3 (Canonical form of an orthogonal map)

Consider the endomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ represented in the standard basis by the following orthogonal matrix in $\mathbb{R}^{(3,3)}$:

$$A = \begin{pmatrix} -1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}.$$

(a) Regard A as a complex matrix via the inclusion $\mathbb{R}^{(3,3)} \subseteq \mathbb{C}^{(3,3)}$, and find its characteristic polynomial over $\mathbb{C}$.

(b) Find a basis of complex eigenvectors $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ of A.

(c) Use this information to find the invariant subspaces of φ regarded again as an endomorphism of $\mathbb{R}^3$. Find an orthonormal basis for $\mathbb{R}^3$ such that in this basis, φ is given by a rotation followed by a reflection.

Exercise 4 (Dual maps)

Let $(V, \langle\cdot,\cdot\rangle^V)$ and $(W, \langle\cdot,\cdot\rangle^W)$ be finite-dimensional euclidean spaces. Recall from Exercise T8.4 that the scalar product of V induces a canonical (i.e., basis-independent) isomorphism $\rho^V : V \to V^*$, where $V^* = Hom(V, \mathbb{R})$ is the *dual space* of V.

$$\rho^V : \quad \begin{aligned} V &\to V^* \\ \mathbf{v} &\mapsto \langle \mathbf{v}, \cdot\rangle^V \end{aligned}$$

where

$$\langle \mathbf{v}, \cdot\rangle^V : \quad \begin{aligned} V &\to \mathbb{R} \\ \mathbf{u} &\mapsto \langle \mathbf{v}, \mathbf{u}\rangle^V \end{aligned}$$

Note that $\rho^W : W \to W^*$ is defined similarly.

(a) Let $\varphi \in Hom(V, W)$ be a linear map. We define the *dual* of φ to be the map $\varphi^* \in Hom(W^*, V^*)$ as follows:

$$\begin{aligned} \varphi^*: \quad & W^* \to V^* \\ & \eta \mapsto \eta \circ \varphi \end{aligned}$$

Note that everything we have defined so far does not depend on a choice of basis. Now let $B_V = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be any basis for V. We define the *dual basis* $B_V^* = (\mathbf{b}_1^*, \ldots, \mathbf{b}_n^*)$ for V^* by the condition $\mathbf{b}_j^*(\mathbf{b}_j) = 0$ for $i \neq j$ and $\mathbf{b}_j^*(\mathbf{b}_j) = 1$ for $i = j$. Similarly, fix a basis $B_W = (\hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_m)$ for W, with associated dual basis B_W^*. Show that the relationship between the matrix representations of φ and φ^* w.r.t. these bases is
$[\![\varphi^*]\!]_{B_V^*}^{B_W^*} = ([\![\varphi]\!]_{B_W}^{B_V})^t$.

(b) What is the status of the map $\varphi^+ := (\rho^V)^{-1} \circ \varphi^* \circ \rho^W$ w.r.t. $\langle \cdot, \cdot \rangle^W$ and $\langle \cdot, \cdot \rangle^V$? Discuss its matrix representations w.r.t. the orthonormal bases B_V and B_W.

(c) In the special case of $V = W = (V, \langle \cdot, \cdot \rangle)$, consider the map $\varphi^+ = (\rho^V)^{-1} \circ \varphi^* \circ \rho^W$ and try to interpret the adjoint of the endomorphism φ in terms of an isomorphic copy of the dual φ^* via canonical identifications of V with V^* via ρ^V.
Analyse the change of basis transformations w.r.t. changes from an onb $B_V (= B_W)$ to another onb $B'_V (= B'_W)$.

Exercise 5 (Positive definiteness and compactness of the unit surface)

(a) Let σ_A be a bilinear form on $\mathbb{R}^n$, which in the standard basis is represented by a symmetric matrix A, whose ijth entry $a_{ij} = \sigma_A(\mathbf{e}_i, \mathbf{e}_j)$. Define the *unit surface*

$$S_A = \{\mathbf{v} \in \mathbb{R}^n : \sigma_A(\mathbf{v}, \mathbf{v}) = 1\}.$$

Suppose that S_A is non-empty. Prove that S_A is compact if and only if σ_A is positive definite.

(b) Let A and B be matrices representing scalar products $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_B$ on $\mathbb{R}^n$. Show that the corresponding norms are equivalent in the sense that there exist positive real numbers m and M satisfying

$$m \langle \mathbf{v}, \mathbf{v} \rangle_A \leqslant \langle \mathbf{v}, \mathbf{v} \rangle_B \leqslant M \langle \mathbf{v}, \mathbf{v} \rangle_A$$

for all $\mathbf{v} \in \mathbb{R}^n$.