Linear Algebra II
Exercise Sheet no. 10

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Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise 1 (Warm-up: self-adjoint maps)

Let V be a finite-dimensional unitary space and $\varphi \in \mathrm{Hom}(V, V)$. Show that the following are equivalent:

(a) φ is self-adjoint.

(b) $\langle \mathbf{v}, \varphi(\mathbf{v}) \rangle \in \mathbb{R}$ for all $\mathbf{v} \in V$.

Hint: Consider $\langle \mathbf{v}+\mathbf{w}, \varphi(\mathbf{v}+\mathbf{w}) \rangle$ and $\langle \mathbf{v}+i\mathbf{w}, \varphi(\mathbf{v}+i\mathbf{w}) \rangle$ for the implication (b)$\Rightarrow$(a).

Exercise 2 (Eigenvalues)

Let V be a finite dimensional vector space and φ, ψ be endomorphisms of V.
Prove that λ is an eigenvalue of $\varphi \circ \psi$ if and only if it is an eigenvalue of $\psi \circ \varphi$.

Hint: It may help to distinguish cases according to whether $\lambda \neq 0$ or $\lambda = 0$.

Extra: Can you give a counterexample in case V is infinite dimensional?

Exercise 3 (Self-adjoint and unitary maps)

Let V be a finite-dimensional unitary space and $\varphi \in \mathrm{Hom}(V, V)$ be a normal endomorphism. Show the following.

(a) φ is self-adjoint if and only if all the eigenvalues of φ are real.

(b) φ is unitary if and only if all the eigenvalues of φ have absolute value 1.

Exercise 4 (Simultaneous diagonalization)

Let V be a finite dimensional unitary space and $\varphi_1, \ldots, \varphi_m$ normal endomorphisms of V that pairwise commute, that is $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$ for all $i, j \in \{1, \ldots, m\}$.

Prove that there exists an orthonormal basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of V consisting of *simultaneous eigenvectors*, that is there are complex numbers λ_{ij} for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, such that

$$\varphi_i(\mathbf{v}_j) = \lambda_{ij}\mathbf{v}_j$$

for all i, j.

(a) Let λ be an eigenvalue of φ_1 and $V_\lambda(\varphi_1) = \{\mathbf{v} \in V \mid \varphi_1(\mathbf{v}) = \lambda\mathbf{v}\}$ the corresponding eigenspace. Prove that

$$\varphi_i(V_\lambda(\varphi_1)) \subseteq V_\lambda(\varphi_1)$$

for all i.

(b) Let λ and μ be two different eigenvalues of φ_i. Show that the corresponding eigenspaces are orthogonal.

(c) Prove now the existence of a basis of V with the desired properties.
Hint: *Induction on m.*

Exercise 5 (Isometries and 'skew-rotations')

We consider the real plane $\mathbb{R}^2$ with the standard scalar product $\langle .,. \rangle$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map that is represented by a rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with respect to some basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. We assume that $\theta \neq 0, \pi$.

Show that φ is an isometry if and only if B is almost an orthonormal basis in the sense that

$$\langle \mathbf{b}_1, \mathbf{b}_2 \rangle = 0 \quad \text{and} \quad \langle \mathbf{b}_1, \mathbf{b}_1 \rangle = \langle \mathbf{b}_2, \mathbf{b}_2 \rangle\,.$$

(So we require the lengths of $\mathbf{b}_1$ and $\mathbf{b}_2$ only to be equal, not to be 1.)