Linear Algebra II
Exercise Sheet no. 9

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Exercise 1 (Warm-up: Isomorphisms of unitary (euclidean) spaces)

(a) Let V and W be euclidean (unitary) spaces of dimension n and $\varphi \in \mathrm{Hom}(V,W)$. Show that the following are equivalent:

 i. φ is an isomorphism of euclidean (unitary) spaces.
 ii. $[\![\varphi]\!]^{B}_{B'} \in O(n)$ for some choice of orthonormal bases B of V and B' of W.
 iii. $[\![\varphi]\!]^{B}_{B'} \in O(n)$ for every orthonormal bases B of V and B' of W.

(b) Conclude that $\varphi \in \mathrm{Hom}(V,V)$ is an orthogonal (unitary) endomorphism of the n-dimensional euclidean (unitary) space V iff $[\![\varphi]\!]^{B}_{B'} \in O(n)$ for some (every) combination of orthonormal bases B and B' of V.
(NB: in one direction this extends Prop. 2.3.15 in the notes.)

Exercise 2 (Composition of two orthogonal projections)

(*Exercise 2.3.4 on page 68 of the notes.*) Let U and W be two subspaces of a finite dimensional euclidean or unitary vector space V, with orthogonal projections π_U and π_W onto U and W, respectively.

Prove that the following statements are equivalent:

(a) π_U and π_W commute.
(b) $\pi_W \circ \pi_U = \pi_{U\cap W}$.
(c) $\pi_W \circ \pi_U$ is an orthogonal projection.
(d) $U = (U \cap W) \oplus (U \cap W^{\perp})$.
(e) $W = (U \cap W) \oplus (U^{\perp} \cap W)$.

Exercise 3 (Endomorphisms that preserve orthogonality)

Let V be a finite dimensional euclidean space. Determine all endomorphisms φ of V that preserve orthogonality, that is for which:

$$\mathbf{v} \perp \mathbf{w} \Rightarrow \varphi(\mathbf{v}) \perp \varphi(\mathbf{w}) \quad \text{for all } \mathbf{v}, \mathbf{w} \in V.$$

Exercise 4 (Jordan normal form and real matrices)

Let $A \in \mathbb{R}^{(n,n)}$ where $n = 2m$ is even. Assume that the characteristic polynomial of A is $p_A = p_0^m$, where $p_0 \in \mathbb{R}[X]$ is an irreducible polynomial of degree 2 in $\mathbb{R}[X]$ (e.g., $p_0 = X^2 + 1$). Hence p_0 splits into linear factors $(\lambda - X)(\overline{\lambda} - X)$ in $\mathbb{C}[X]$, with $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(a) Show that if $\mathbf{v}$ is a generalised eigenvector for λ with height k, then $\overline{\mathbf{v}}$ is a generalised eigenvector for $\overline{\lambda}$ with height k, and $[\![\mathbf{v}]\!] \cap [\![\overline{\mathbf{v}}]\!] = 0$. (*Hint.* Use Lemma 1.5.6.)

(b) Show that A is similar to a real matrix $K \in \mathbb{R}^{(n,n)}$ composed of just three kinds of (2×2)-blocks: $\mathbf{0} \in \mathbb{R}^{(2,2)}$, $E_2 \in \mathbb{R}^{(2,2)}$ and some $A_0 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathbb{R}^{(2,2)}$ with $b \neq 0$, where A_0 occurs along the diagonal, E_n and $\mathbf{0}$ immediately above the diagonal and just $\mathbf{0}$ everywhere else (a "block Jordan normal form").
Hint. Put A into Jordan normal form over $\mathbb{C}$ w.r.t. basis consisting of complex conjugate vector pairs; then combine such pairs to find a real basis.

(c) Give examples of $A_k \in \mathbb{R}^{(6,6)}$ with characteristic polynomial $(X^2+1)^3$ and minimal polynomials $q_{A_k} = (X^2+1)^k$ for $k = 1, 2, 3$.