Linear Algebra II
Exercise Sheet no. 8

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Exercise 1 (Warm-up: orthogonal complement and orthogonal projection)

Let V be a euclidean or unitary vector space of finite dimension, U be a subspace of V and $\pi_U : V \to U$ be the orthogonal projection onto U. Check the following facts.

(a) $U^{\perp}$ is a subspace of V.

(b) If B is a basis of U, then $U^{\perp} = \{\mathbf{v} \in V \mid \mathbf{v} \perp B\}$.

(c) π_U is linear, surjective and $ker(\pi_U) = U^{\perp}$.

(d) $\pi_U \circ \pi_U = \pi_U$.

(e) For any subspace W of V,

$$\pi_U^{-1}(W) = (U \cap W) \oplus U^{\perp}.$$

(f) If $B = (\mathbf{v}_1, \dots, , \mathbf{v}_n)$ is an orthonormal basis of U, then

$$\pi_U(\mathbf{v}) = \sum_{i=1}^{n} \langle \mathbf{v}_i, \mathbf{v} \rangle \mathbf{v}_i.$$

Exercise 2 (Orthogonal complements)

(*Exercise 2.3.5 on page 68 of the notes.*) Let V be a euclidean or unitary vector space of finite dimension. Moreover, let U, U_1, U_2 be subspaces of V. Prove the following facts.

(a) $U_1 \subseteq U_2$ implies $U_2^{\perp} \subseteq U_1^{\perp}$.

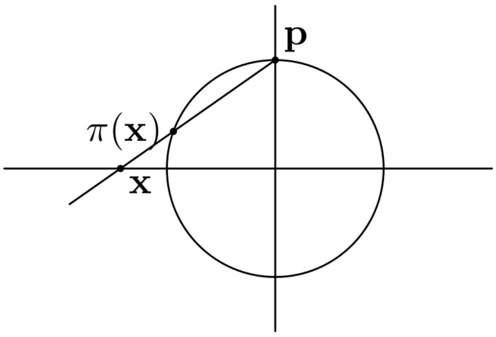
(b) $(U_1 + U_2)^{\perp} = U_1^{\perp} \cap U_2^{\perp}$.

(c) $(U^{\perp})^{\perp} = U$.

(d) $(U_1 \cap U_2)^{\perp} = U_1^{\perp} + U_2^{\perp}$.

Exercise 3 (Stereographic projection)

Let $E \subseteq \mathbb{R}^3$ be the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ and let $S \subseteq \mathbb{R}^3$ be the sphere with radius 1 and centre $\mathbf{0}$. We denote the north pole of S by $\mathbf{p} := \mathbf{e}_3$ and we set $S_* := S \setminus \{\mathbf{p}\}$.



We define a map $\pi : E \to S_*$ by letting $\pi(\mathbf{x})$ be the point of intersection between S_* and the line passing through $\mathbf{p}$ and $\mathbf{x}$.

(a) Give an explicit formula for π, i.e., find functions $f(x,y)$, $g(x,y)$, and $h(x,y)$ such that $\pi(x,y,0) = (f(x,y), g(x,y), h(x,y))$.

(b) Prove that $\pi : E \to S_*$ is a bijection.

(c) Let $C \subseteq S$ be a circle, i.e., the intersection of S with a plane given by an equation of the form $ax + by + cz = d$. Prove that the pre-image $\pi^{-1}[C]$ is either also a circle or a line.

(d) Let $c : \mathbb{R} \to E$ be a line with parametric description $x\mathbf{e}_1 + t\mathbf{v}$, $t \in \mathbb{R}$, where $\mathbf{v} = (\cos\alpha, \sin\alpha, 0)$. Note that c intersects the $\mathbf{e}_1$-axis in the point $x\mathbf{e}_1$ under the angle α. Prove that the image of c under π, i.e., the curve $\pi \circ c : \mathbb{R} \to S_*$, intersects the great circle $\{(u,0,v) \in S_* : u^2 + v^2 = 1\}$ under the same angle α. (This implies that π preserves angles. Such maps are called *conformal.*)

(*Hint.* Find the angle between the tangent vectors of the two curves. The tangent vector of a curve c at the point $c(t_0)$ is given by its derivative $\frac{\mathrm{d}}{\mathrm{d}t}c\Big|_{t_0}$.)

Exercise 4 (Characterisations of orthogonal projections)

(*Exercise 2.3.2 on page 68 of the notes.*) Let φ be an endomorphism of a finite dimensional euclidean or unitary vector space V.

Show the equivalence of the following:

(a) φ is an orthogonal projection.

(b) $\varphi \circ \varphi = \varphi$ and $\ker(\varphi) \perp \mathrm{image}(\varphi)$.

(c) $\varphi \circ \varphi = \varphi$ and $\mathbf{v} - \varphi(\mathbf{v}) \perp \varphi(\mathbf{v})$ for all $\mathbf{v} \in V$.

(d) $\mathbf{v} - \varphi(\mathbf{v}) \perp \mathrm{image}(\varphi)$ for all $\mathbf{v} \in V$.

Exercise 5 (More on orthogonal projections)

(*Exercise 2.3.3 on page 68 of the notes.*) Show that the orthogonal projections of an n-dimensional euclidean or unitary vector space V are precisely those endomorphisms φ of V that are represented w.r.t. a suitable orthonormal basis by a diagonal matrix with ones and zeroes on the diagonal.