Linear Algebra II
Exercise Sheet no. 7

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Exercise 1 (Warm up: the trace)

Recall Exercise E4.3 about the trace.

Let $V := \mathbb{R}^{(n,n)}$ be the $\mathbb{R}$-vector space of all real $n \times n$ matrices and let $S \subseteq V$ be the subspace consisting of all symmetric matrices (i.e., all matrices A with $A^t = A$). For $A, B \in V$, we define

$$\langle A, B\rangle := \mathrm{Tr}(AB),$$

where the *trace* $\mathrm{Tr}(A)$ of a matrix $A = (a_{ij})$ is defined as

$$\mathrm{Tr}(A) := \sum_{i=1}^{n} a_{ii}.$$

(a) Show that $\langle .\,, .\rangle$ is bilinear.

(b) Show that $\langle .\,, .\rangle$ is a scalar product on S.

Exercise 2 (Cauchy-Schwarz and triangle inequalities)

(a) (Exercise 2.1.4 on page 60 of the notes)
Let $(V, \langle .\,, .\rangle)$ be a euclidean or unitary vector space. Show that equality holds in the Cauchy-Schwarz inequality, i.e., we have $\|\langle \mathbf{v}, \mathbf{w}\rangle\| = \|\mathbf{v}\| \cdot \|\mathbf{w}\|$, if, and only if, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

(b) (Exercise 2.1.5 on page 60 of the notes)
Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be pairwise distinct vectors in a euclidean or unitary vector space $(V, \langle .\,, .\rangle)$, and write $\mathbf{a} := \mathbf{v} - \mathbf{u}$, $\mathbf{b} := \mathbf{w} - \mathbf{v}$. Show that equality holds in the triangle inequality

$$d(\mathbf{u}, \mathbf{w}) = d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}), \text{ or, equivalently, } \|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|,$$

if, and only if, $\mathbf{a}$ and $\mathbf{b}$ are *positive real* scalar multiples of each other (geometrically: $\mathbf{v} = \mathbf{u} + s(\mathbf{w} - \mathbf{u})$ for some $s \in (0, 1) \subseteq \mathbb{R}$).

Exercise 3 (Orthogonal matrices)

We consider real $n \times n$ matrices. Set

$$\mathrm{O}(n) := \{A \in \mathbb{R}^{(n,n)} \mid A^t A = E_n\}.$$

Show that $\mathrm{O}(n)$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

Exercise 4 (Orthogonal vectors)

Let V be a euclidean or unitary space and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of non-null pairwise orthogonal vectors.

(a) Show that S is linearly independent.

(b) Let $\mathbf{u} \in V$. Show that the vector

$$\mathbf{w} := \mathbf{u} - \sum_{i=1}^{n} \frac{\langle \mathbf{v}_i, \mathbf{u}\rangle}{\langle \mathbf{v}_i, \mathbf{v}_i\rangle} \mathbf{v}_i$$

is orthogonal to S. Note that $\sum_{i=1}^{n} \frac{\langle \mathbf{v}_i, \mathbf{u}\rangle}{\langle \mathbf{v}_i, \mathbf{v}_i\rangle} \mathbf{v}_i$ is the orthogonal projection of $\mathbf{w}$ on $\mathrm{span}(S)$.

(c) **[Parseval's identity]** Suppose that V is finite dimensional and that S is an othornormal basis of V. Show that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{v}_i \rangle \langle \mathbf{v}_i, \mathbf{w} \rangle \quad \text{for all } \mathbf{v}, \mathbf{w} \in V\,.$$

(d) **[Bessel's inequality]** Suppose that V is euclidean and S is orthonormal. Show that

$$\sum_{i=1}^{n} \langle \mathbf{v}_i, \mathbf{u} \rangle^2 \leq \|\mathbf{u}\|^2 \quad \text{for all } \mathbf{u} \in V\,.$$

Exercise 5 (Jordan normal form for describing processes)

Suppose that we use vectors $\mathbf{s}_n \in \mathbb{R}^3$ to describe the state of a 3-dimensional system at step $n \in \mathbb{N}$ (for example, the position of a particle in space). The evolution of the system from stage n to $n+1$ is given by

$$\mathbf{s}_{n+1} = A\mathbf{s}_n\,, \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 & -1 \\ -4 & 3 & 0 \\ 14 & -5 & 5 \end{pmatrix}.$$

(a) Use a transformation of the given A into Jordan normal form in order to get a feasible formula for $\mathbf{s}_n$, as a function of the index n and the initial state $\mathbf{s}_0$.

(b) Compute $\mathbf{s}_{100}$ for $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$.