Linear Algebra II
Exercise Sheet no. 6

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Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise 1 (Warm-up: possible Jordan normal forms)

Let $\varphi : V \to V$ be an endomorphism of a finite dimensional $\mathbb{C}$-vector space V. Which of the following situations can occur?

(a) i. V is 6-dimensional, the minimal polynomial of φ is $(X-2)^5$, and the eigenspace of 2 has dimension 3.
   ii. V is 6-dimensional, the minimal polynomial of φ is $(X-2)(X-3)^2$, and the eigenspace of 2 has dimension 3.

(b) i. φ has minimal polynomial $(X-2)^4$ and there is a vector $\mathbf{v} \in V$ with height 3.
   ii. φ has minimal polynomial $(X-2)^4$ and there is a vector $\mathbf{v} \in V$ with height 6.
   iii. φ has minimal polynomial $(X-2)^4$, but no vector in V has height 3.

(c) i. φ has characteristic polynomial $(X-2)^6$ and $\varphi^2 - \varphi - \mathrm{id} = \mathbf{0}$.
   ii. $\varphi^2 - \varphi - 2\mathrm{id} = \mathbf{0}$ and φ has eigenvalues that are not real.

(d) i. V has a φ-invariant subspace of dimension 5, 2 is the only eigenvalue of φ, but there is no $\mathbf{v} \in V$ with $\dim(\llbracket \mathbf{v} \rrbracket) = 5$.
   ii. 2 is the only eigenvalue of φ, $V = \llbracket \mathbf{v} \rrbracket \oplus \llbracket \mathbf{b} \rrbracket$ with $\dim(\llbracket \mathbf{v} \rrbracket) = 5$, but the Jordan normal form for φ contains no block of size 5.

(e) i. V can be written as the direct sum of two φ-invariant subspaces of dimension 4, but there is no Jordan block of size greater than 3 in the Jordan normal form for φ.
   ii. V can be written as the direct sum of two φ-invariant subspaces of dimension 4, and in the Jordan normal form of φ there is a Jordan block of size 5.

Exercise 2 (Commuting matrices and simultaneous diagonalization)

(a) Let M_1 and M_2 be square matrices over $\mathbb{F}$, and let q_{M_1} and q_{M_2} be the corresponding minimal polynomials. Show that the minimal polynomial of the block matrix

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

is the least common multiple of q_{M_1} and q_{M_2}. (Clearly this observation generalises to block matrices with an arbitrary number of blocks).

(b) Show that M is diagonalizable if and only if both M_1 and M_2 are diagonalizable.

(c) Let A and B be diagonalizable $n \times n$ matrices over $\mathbb{F}$ that commute with each other, i.e., $AB = BA$.
   i. Show that any eigenspace of A is invariant under B.
   ii. Show that A and B are *simultaneously diagonalizable*, i.e., there exists a matrix C such that $C^{-1}AC$ and $C^{-1}BC$ are both diagonal matrices.

Exercise 3 (Computing the Jordan normal form)

Let

$$A := \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & -1 & -3 & -2 \\ -2 & 3 & 5 & 2 \\ -1 & 2 & 2 & 3 \end{pmatrix}.$$

Find a regular matrix S and a matrix J in Jordan normal form such that $A = SJS^{-1}$.

Hint. The characteristic polynomial of A is $p_A = (2-X)^4$.

Exercise 4 (Exponential function for matrices)
Let

$$J_\lambda := \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \in \mathbb{C}^{n\times n}$$

be a Jordan block with eigenvalue λ. For an arbitrary matrix A, we define

$$e^A := \sum_{i=0}^{\infty} \frac{A^i}{i!}.$$

(a) Compute J_0^k.

(b) Compute J_λ^k. *Hint.* Use the decomposition $J_\lambda = \lambda E_n + J_0$.

For the following we leave aside all the convergence issues. It is indeed safe here, but not part of linear algebra.

(c) Suppose that A and B are matrices with $AB = BA$. Show that $e^{A+B} = e^A e^B$.

(d) Show that $e^{S^{-1}AS} = S^{-1}e^A S$, for an arbitrary matrix A and an invertible one S.

(e) Prove that

$$e^{J_\lambda} = e^\lambda \sum_{i=0}^{n-1} \frac{J_0^i}{i!}.$$

Exercise 5 (Square roots)

(a) Let $a_0, \dots, a_{n-1} \in \mathbb{C}$ and let N be the $n \times n$ matrix $\begin{pmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & \dots & 0 \end{pmatrix}$. When is $(\sum_{i=0}^{n-1} a_i N^i)^2$ a Jordan block?

(b) Deduce a sufficient condition for $dE_n + N \in \mathbb{C}^{(n,n)}$ to have a square root.

(c) Deduce a sufficient condition for complex matrices to have complex square roots.

Remark: using techniques from Lie group theory, which combine differential geometry, topology and group theory, one can also obtain that the exponential map on matrices, $A \mapsto e^A$, is a surjection of $\mathbb{C}^{(n,n)}$ onto $GL_n(\mathbb{C})$. It follows that the equality $[e^{\frac{1}{2}A}]^2 = e^A$ yields square roots for any regular matrix.