Linear Algebra II
Exercise Sheet no. 5

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Exercise 1

(a) Consider 2×2-matrices over the complex numbers. Why does their minimal polynomial determine their characteristic polynomial? Is the same true for 3×3-matrices?

(b) Find two 2×2-matrices that are not similar, but have the same characteristic polynomial.

(c) Show that any two 2×2-matrices with the same minimal polynomial are similar in $\mathbb{C}^{(2,2)}$. Is the same true in $\mathbb{R}^{(2,2)}$?

(d) Discuss necessary and sufficient conditions (also in terms of the determinant, the trace, and the minimal and characteristic polynomial of a matrix) for the similarity of two matrices. Use these criteria to split the following 9 matrices into equivalence classes w.r.t. similarity.

$$A_1 = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 3 & 2 \\ 6 & 8 & 7 \end{pmatrix} \quad A_2 = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 7 & 2 \\ 2 & 8 & 6 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad A_5 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad A_6 = \begin{pmatrix} 2 & 4 & 3 \\ 3 & 1 & 2 \\ 8 & 6 & 7 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 4 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix} \quad A_8 = \begin{pmatrix} 2 & 5 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{pmatrix}, \quad A_9 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 2 (Endomorphisms and bases)

Let $\varphi \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^3)$ be an endomorphism of $\mathbb{R}^3$ that, for some $\lambda \in \mathbb{R}$, is represented by the matrix

$$A_\lambda := \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

(a) Check that the third basis vector in a basis B giving rise to A_λ as $A_\lambda = [\![\varphi]\!]_B^B$ must be in $\ker(\varphi - \lambda\mathrm{id})^3 \setminus \ker(\varphi - \lambda\mathrm{id})^2$.

(b) Describe in words which properties of φ guarantee that $[\![\varphi]\!]_B^B = A_\lambda$ for some basis B (for instance, in terms of eigenvalues, eigenvectors, the minimal polynomial, or the characteristic polynomial).

(c) For fixed φ (and λ), describe the set of all bases $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ for which $[\![\varphi]\!]_B^B = A_\lambda$.
Hint. Use φ to express $\mathbf{b}_1$ in terms of $\mathbf{b}_2$ and $\mathbf{b}_2$ in terms of $\mathbf{b}_3$, and determine the possible choices for $\mathbf{b}_3$.

(d) For $\lambda = 0$, what does the condition that $[\![\varphi]\!]_B^B = A_0$, for some basis B, tell us about dimensions of and the relationship between $\mathrm{Im}(\varphi)$ and $\ker(\varphi)$? What are the invariant subspaces?

Exercise 3 (Nilpotent endomorphisms)

Recall that an endomorphism $\varphi : V \to V$ is *nilpotent* if there is some $k \in \mathrm{N}$ such that $\varphi^k = 0$. The minimal such k is called the *index* of φ.

(a) Suppose that V is $\mathrm{Pol}_n(\mathbb{R})$ the $\mathbb{R}$-vector space of all polynomial functions of degree up to n. Show that the usual differential operator $\partial : V \to V : f \mapsto f'$ is nilpotent of index $n+1$.

Suppose that $\varphi : V \to V$ is nilpotent with index k.

(b) Show that $q_\varphi = X^k$.

(c) Show that, for every $\mathbf{v} \in V$, $W := \mathrm{span}(\mathbf{v}, \varphi(\mathbf{v}), \ldots, \varphi^{k-1}(\mathbf{v}))$ is an invariant subspace.

(d) Let W be the subspace from (iii) where we additionally assume that $\varphi^{k-1}(\mathbf{v}) \neq \mathbf{0}$. Show that the restriction φ_0 of φ to W is nilpotent with index k.

(e) Suppose that V has dimension k. Show that there is some basis B such that

$$[\![\varphi]\!]_B^B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & \cdots & & 0 \end{pmatrix}.$$

Exercise 4 (Characteristic and minimal polynomial)

Let $A \in \mathbb{F}^{(n,n)}$ have the characteristic polynomial p_A and the minimal polynomial $q_A = X^r + \sum_{i=0}^{r-1} c_i X^i$.

(a) Let $B_0, B_1, \ldots, B_r$ be defined as below.

$$\begin{aligned} B_0 &:= E_n \\ B_1 &:= A + c_{r-1}E_n \\ B_2 &:= A^2 + c_{r-1}A + c_{r-2}E_n \\ \cdots & \\ B_{r-1} &:= A^{r-1} + c_{r-1}A^{r-2} + \cdots + c_1 E_n \\ B_r &:= A^r + c_{r-1}A^{r-1} + \cdots + c_0 E_n \end{aligned}$$

Let $B(X) := X^{r-1}B_0 + X^{r-2}B_1 + \cdots + XB_{r-2} + B_{r-1}$ and show that $(XE_n - A)B(X) = q_A(XE_n)$.

(b) Use part (a) to show that p_A divides $(q_A)^n$.

(c) Use part (b) to show that p_A and q_A have the same irreducible factors.