Linear Algebra II
Exercise Sheet no. 4

TECHNISCHE UNIVERSITÄT DARMSTADT

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Exercise 1 (Warm-up)
Prove the Cayley-Hamilton Theorem for 2×2 matrices by direct computation.

Exercise 2 (Eigenvalues)
Let p be a polynomial in $\mathbb{F}[X]$ and $A \in \mathbb{F}^{(n,n)}$. Show that if λ is an eigenvalue of A, then $p(\lambda)$ is an eigenvalue of the matrix $p(A)$.

Exercise 3 (Trace)
Recall that $\operatorname{tr}(A) = \sum_{i=1}^n a_{ii}$ is the *trace* of an $n \times n$-matrix $A = (a_{ij}) \in \mathbb{F}^{(n,n)}$.

(a) Show that for any matrices $A, B \in \mathbb{F}^{(n,n)}$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Use this to show that similar matrices have the same trace.

(b) How does the characteristic polynomial p_A of a matrix A determine $\operatorname{tr}(A)$ and $\det(A)$? From this, conclude (again) that the trace is invariant under similarity (as is the determinant, of course).

(c) Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Show that $\operatorname{tr}(ABC) \neq \operatorname{tr}(ACB)$. Therefore the trace is not invariant under arbitrary permutations of products of matrices.

Exercise 4 (Characteristic polynomial)

(a) Determine the characteristic polynomial p_A of the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & a_0 \\ 1 & 0 & 0 & \dots & 0 & 0 & a_1 \\ 0 & 1 & 0 & \dots & 0 & 0 & a_2 \\ 0 & 0 & 1 & \dots & 0 & 0 & a_3 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 & a_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1} \end{pmatrix}, \quad \text{with } n \geq 1.$$

Hint: expand the determinant along the last column.

(b) Show that every polynomial $p \in \mathbb{F}[X]$ of degree $n \geq 1$ with leading coefficient $(-1)^n$ occurs as a characteristic polynomial of a matrix $A \in \mathbb{F}^{(n,n)}$.

Exercise 5 (Polynomials over $\mathbb{F}_2$)

(a) Show that in $\mathbb{F}_2[X]$ any non-linear polynomial with an odd number of powers X^i for $i \geq 1$ (with or without the constant term 1) is reducible.

(b) Find in $\mathbb{F}_2[X]$ all irreducible polynomials of degree 3 and 4.