Linear Algebra II
Exercise Sheet no. 3

TECHNISCHE UNIVERSITÄT DARMSTADT

SS 2011
April 29, 2011

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise 1 (Warm-up: Multiple Zeroes)

For a polynomial $p = \sum_{i=0}^{n} a_i X^i \in \mathbb{F}[X]$ define its *formal derivative* p' by

$$p' := \sum_{i=1}^{n} i a_i X^{i-1}.$$

(a) Check that the usual product rule for differentiaton applies to the formal derivative of polynomials considered here!

(b) Let α be a zero of P. Show the equivalence of the following:

i. α is a multiple zero of p. (In other words, $(X-\alpha)^2$ divides p.)

ii. α is a zero of p'.

iii. α is a zero of $\gcd(p, p')$.

Exercise 2 (Commutative subrings of matrix rings)

Let $A \in \mathbb{F}^{(n,n)}$ be an $n \times n$ matrix over a field $\mathbb{F}$. Let $R_A \subseteq \mathbb{F}^{(n,n)}$ be the subring generated by A, which consists of all linear combinations of powers of A.

(a) Prove that R_A is a commutative subring of $\mathbb{F}^{(n,n)}$.

(b) Consider the evaluation map $\tilde{\ }: \mathbb{F}[X] \to R_A$ defined by $\tilde{p} = \sum_i^n a_i A^i$ for $p = \sum_i^n a_i X^i$. Show that this map is a ring homomorphism. Is it surjective? Injective?

Hint: By forgetting about the multiplicative structure, we may regard $\mathbb{F}[X]$ and R_A as vector spaces over $\mathbb{F}$, and we may regard $\tilde{\ }$ as a vector space homomorphism. Do $\mathbb{F}[X]$ and R_A have the same dimension as $\mathbb{F}$-vector spaces?

Exercise 3 (The Euclidean algorithm revisited)

Recall the Euclidean algorithm from Exercise Sheet 2. In particular, given natural numbers a, b, we normalise so that $d_1 = \min\{a, b\}$ and $d_0 = \max\{a, b\}$. In each step, we divide with remainder, obtaining $d_{k-1} = q_k d_k + d_{k+1}$. At the end of this procedure $d_{k+1} = 0$, and $d_k = \gcd(a, b)$.

(a) Let k be the number of steps needed to compute $\gcd(a_0, b_0)$ in this way. Consider the matrix $M \in \mathbb{Z}^{(2,2)}$ given by

$$M = \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & q_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & q_k \end{pmatrix}.$$

Show that M is regular and that M^{-1} is again a matrix over $\mathbb{Z}$. Compute $M^{-1}\begin{pmatrix} d_1 \\ d_0 \end{pmatrix}$.

(b) Interpret the entries in second row of M^{-1} in terms of $\gcd(d_0, d_1)$.

(c) Recall that the *least common multiple* $\mathrm{lcm}(d_0, d_1)$ is an integer z characterized by the following properties:

i. $d_0 | z$ and $d_1 | z$.

ii. If a is any integer for which $d_0 | a$ and $d_1 | a$, then $z | a$.

Interpret the entries in the first row of M^{-1} in terms of $\mathrm{lcm}(d_0, d_1)$.

Exercise 4 (Polynomial factorisation and diagonalisation)

Consider the following polynomials in $\mathbb{F}[X]$ for $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$:

$$p_1 = X^3 - 2, \qquad p_2 = X^3 + 4X^2 + 2X, \qquad p_3 = X^3 - X^2 - 2X + 2.$$

(a) Which of these polynomials are irreducible in $\mathbb{F}[X]$?

(b) Which of these polynomials decompose into linear factors over $\mathbb{F}[X]$?

(c) Suppose p_i is the characteristic polynomial of a matrix $A_i \in \mathbb{F}^{(3,3)}$. Which of the A_i is diagonalisable over $\mathbb{F}$?